On the Existence of Generalized Parking Spaces for Complex Reflection Groups

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Plan:

- Generalized parking spaces
- Main theorem
- Proof for the symmetric groups

  — Greatest common divisors of specialized Schur functions

Generalized Parking Spaces

Parking Functions

A parking function of length n is a sequence $(a_1, a_2, \cdots, a_n)$ of positive integers satisfying

- $a_i \in \{1, 2, \cdots, n\}$, and
- $\#\{i : a_i \le k\} \ge k$ for $k = 1, 2, \cdots, n$.

We put

$$\mathrm{PF}_n = \text{the set of parking functions of length } n.$$

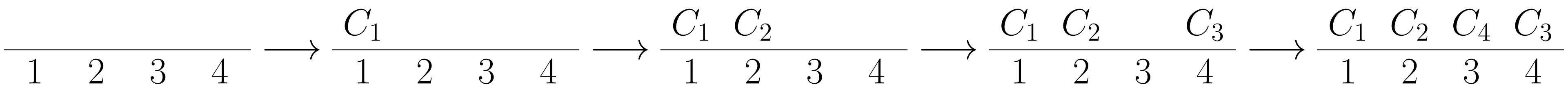
Example

$$\mathrm{PF}_2 = \{\, 11,\ 12,\ 21 \,\},$$
$$\mathrm{PF}_3 = \left\{ \begin{matrix} 111, & 112, & 121, & 211, & 113, & 131, & 311, & 122 \\ 212, & 221, & 123, & 132, & 213, & 231, & 312, & 321 \end{matrix} \right\}.$$

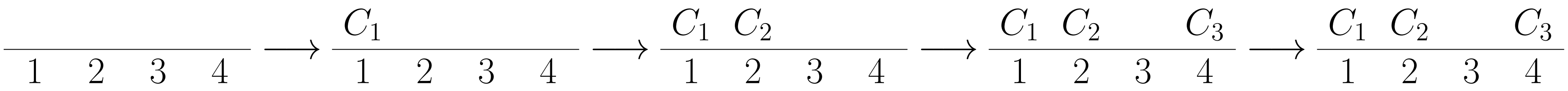
Imagine that there are n cars $C_1, C_2, \cdots, C_n$ and n parking spaces $1, 2, \cdots, n$ in a one-way street. Car C_i prefers the parking space a_i and approaches its preferred parking space.

- If it is free, then C_i parks there.
- If it is occupied, then C_i parks in the next available space if possible.

Example If $n = 4$ and $(a_1, a_2, a_3, a_4) = (1, 1, 4, 2)$, then



If $n = 4$ and $(a_1, a_2, a_3, a_4) = (1, 1, 4, 4)$, then



It is not hard to see that the sequence $(a_1, \cdots, a_n)$ is a parking function if and only if all cars can park.

Symmetric group action on parking functions

The symmetric group $\mathfrak{S}_n$ acts on the set PF_n by permuting entries:

$$\sigma \cdot (a_1, \cdots, a_n) = (a_{\sigma(1)}, \cdots, a_{\sigma(n)}) \quad (\sigma \in \mathfrak{S}_n).$$

It is known that there exists an $\mathfrak{S}_n$-equivariant bijection

$$\mathrm{PF}_n \longrightarrow \{x \in (\mathbb{Z}/(n+1)\mathbb{Z})^n : x_1 + \cdots + x_n = 0\}.$$

and that the corresponding permutation character is given by

$$\varphi(\sigma) = (n+1)^{l(\mathrm{type}(\sigma))-1} \quad (\sigma \in \mathfrak{S}_n),$$

where $\mathrm{type}(\sigma)$ is the cycle type of σ.

More generally, given a positive integer k, we consider the class function on $\mathfrak{S}_n$ defined by

$$\varphi_k(\sigma) = k^{l(\mathrm{type}(\sigma))-1} \quad (\sigma \in \mathfrak{S}_n).$$

Question When is φ_k the character of some representation of $\mathfrak{S}_n$?

It is not hard to show that, if k is relatively prime to n, then φ_k is the permutation character on

$$\{x \in (\mathbb{Z}/k\mathbb{Z})^n : x_1 + \cdots + x_n = 0\}.$$

Generalization to Complex Reflection Groups

Let V be a finite dimensional complex vector space. A complex reflection is an element $g \in \mathbf{GL}(V)$ with $\dim V^g = \dim V - 1$, where $V^g = \{v \in V : gv = v\}$. A complex reflection group is a subgroup $W \subset \mathbf{GL}(V)$ which is generated by complex reflections.

Example The symmetric group $\mathfrak{S}_n$ is realized as an irreducible complex reflection group acting on

$$V = \{x \in \mathbb{C}^n : x_1 + \cdots + x_n = 0\}.$$

In this setting, we have

$$l(\mathrm{type}(\sigma)) - 1 = \dim V^\sigma \quad (\sigma \in \mathfrak{S}_n).$$

Let W be a finite complex reflection group acting on V. Let k be a positive integer and consider the class function φ_k on W given by

$$\varphi_k(w) = k^{\dim V^w} \quad (w \in W),$$

where V^w is the fixed-point subspace of w.

Question When is φ_k the character of a representation of W?

We call such a W-module a generalized k-parking space for W.

Example

- The complex vector space $\mathbb{C}[\mathrm{PF}_n]$ with basis PF_n is an $(n+1)$-parking space for $\mathfrak{S}_n$.
- If W is a Coxeter group and $k = h+1$, where h is the Coxeter number, then a certain finite dimensional module over the corresponding rational Cherednik algebra provides a generalized $(h+1)$-parking space for W.

q-Analogue

Let q be an indeterminate. A natural q-analogue of the class function φ_k is given by

$$\widetilde{\varphi}_k(w) = \frac{\det_V(1 - q^k w)}{\det_V(1 - qw)} \quad (w \in W).$$

For example, if W is a Coxeter group and $k \equiv 1 \bmod h$, where h is the Coxeter number, then this class function appears as the $W \times \mathfrak{sl}_2$ character of a certain finite dimensional module over the rational Cherednik algebra associated to W.

Question When is $\widetilde{\varphi}_k$ the graded character of a graded representation of W? In other words, when is there a graded W module $U = \bigoplus_i U_i$ such that

$$\widetilde{\varphi}_k = \sum_i \mathrm{char}(U_i) q^i \quad ?$$

Main Theorem

Generalized q-Catalan numbers

Let W be a finite complex reflection group acting on V. Let $(d_1, \ldots, d_r)$ and $(d_1^*, \ldots, d_r^*)$ be the degrees and the codegrees of W respectively. Then we define the generalized q-Catalan number $\mathrm{Cat}_k(W, q)$ and the generalized dual q-Catalan number $\mathrm{Cat}_k^*(W, q)$ as follows:

$$\mathrm{Cat}_k(W, q) = \prod_{i=1}^{r} \frac{[k + d_i - 1]_q}{[d_i]_q},$$

$$\mathrm{Cat}_k^*(W, q) = q^N \prod_{i=1}^{r} \frac{[k - d_i^* - 1]_q}{[d_i]_q},$$

where N is the number of reflecting hyperplanes and

$$[m]_q = \frac{1 - q^m}{1 - q}.$$

Example If $W = \mathfrak{S}_n$ is the symmetric group, then the degrees and codegrees are

$$(d_i) = (2, 3, \dots, n-1, n) = (d_i^*).$$

If $k = n+1$ (the Coxeter number of $\mathfrak{S}_n$ plus 1), then we have

$$\mathrm{Cat}_{n+1}(\mathfrak{S}_n, q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

which is a q-analogue of the Catalan numbers.

Example In general, if W is a complex reflection group and h is the Coxeter number, then

$$\mathrm{Cat}_{h+1}(W, q) = \prod_{i=1}^{r} \frac{[h + d_i]_q}{[d_i]_q}$$

is a q-analogue of the W-Coxeter number.

Main Theorem

Let W be a finite irreducible complex reflection group. For a positive integer k, the following are equivalent:

(1) $\widetilde{\varphi}_k$ is the graded character of a graded representation of W.

(2) $\mathrm{Cat}_k^*(W, q)$ is a polynomial in q.

(3) k satisfies the following congruence condition:

group	condition on k
$\mathfrak{S}_n$	$\gcd(n, k) = 1$
$G(m, p, n)$	$\begin{cases} k \equiv \pm 1 \bmod m & \text{if } n = 2 \text{ and } p = m \\ k \equiv 1 \bmod m & \text{otherwise} \end{cases}$
C_m	$k \equiv 1 \bmod m$

where $G(m, p, n)$ is a normal subgroup of index p in $G(m, 1, n) = (C_m)^n \rtimes \mathfrak{S}_n$, and C_m is the cyclic group of order m.

group	condition on k
4	$k \equiv 1, 3 \bmod 6$
5	$k \equiv 1 \bmod 6$
6	$k \equiv 1, 9 \bmod 12$
7	$k \equiv 1 \bmod 12$
8	$k \equiv 1, 5 \bmod 12$
9	$k \equiv 1, 17 \bmod 24$
10	$k \equiv 1 \bmod 12$
11	$k \equiv 1 \bmod 24$
12	$k \equiv 1, 11, 17, 19 \bmod 24$
13	$k \equiv 1, 17 \bmod 24$
14	$k \equiv 1, 19 \bmod 24$
15	$k \equiv 1 \bmod 24$
16	$k \equiv 1, 11 \bmod 30$
17	$k \equiv 1, 41 \bmod 60$
18	$k \equiv 1 \bmod 30$
19	$k \equiv 1 \bmod 60$
20	$k \equiv 1, 19 \bmod 30$

group	condition on k
21	$k \equiv 1, 49 \bmod 60$
22	$k \equiv 1, 29, 41, 49 \bmod 60$
23	$k \equiv 1, 5, 9 \bmod 10$
24	$k \equiv 1, 9, 11 \bmod 14$
25	$k \equiv 1 \bmod 6$
26	$k \equiv 1 \bmod 6$
27	$k \equiv 1, 19, 25 \bmod 30$
28	$k \equiv 1, 5 \bmod 6$
29	$k \equiv 1, 9, 13, 17 \bmod 20$
30	$k \equiv 1, 11, 19, 29 \bmod 30$
31	$k \equiv 1, 13, 17, 29, 37, 41, 49, 53 \bmod 60$
32	$k \equiv 1, 7, 13, 19 \bmod 30$
33	$k \equiv 1 \bmod 6$
34	$k \equiv 1, 13, 19, 25, 31, 37 \bmod 42$
35	$k \equiv 1, 5 \bmod 6$
36	$k \equiv 1, 5 \bmod 6$
37	$k \equiv 1, 7, 11, 13, 17, 19, 23, 29 \bmod 30$

Main Theorem

Let W be a finite irreducible complex reflection group. For a positive integer k, the following are equivalent:

(1) $\widetilde{\varphi}_k$ is the graded character of a graded representation of W.

(2) $\mathrm{Cat}^*_k(W, q)$ is a polynomial in q.

(3) k satisfies the following congruence condition:

e.g., $\gcd(n, k) = 1$ for $W = \mathfrak{S}_n$.

Except for the dihedral groups, the above conditions are equivalent to the following:

(4) φ_k is the character of a representation of W.

(5) φ_k is the character of a permutation representation of W.

Remark If $W = G(m,m,2) = I_2(m)$ is the dihedral group of order $2m$, then we have

$$\widetilde{\varphi}_k \text{ is the graded character of a graded representation of } W \iff k \equiv \pm 1 \bmod m$$

and

$$\varphi_k \text{ is the character of a representation of } W \iff \begin{cases} k = 1, \text{ or} \\ k \geq m-1 \text{ and } k^2 \equiv 1 \begin{cases} \bmod 2m & \text{if } m \text{ is even,} \\ \bmod m & \text{if } m \text{ is odd.} \end{cases} \end{cases}$$

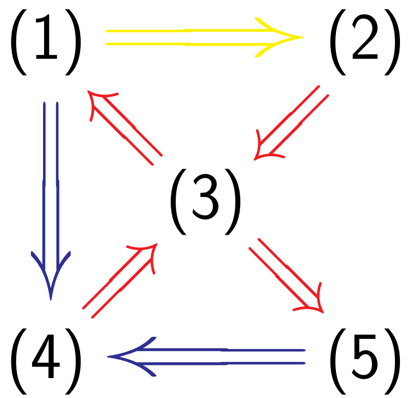
Known cases

- Haiman proved the equivalence of (1), (2) and (3) for the symmetric group $\mathfrak{S}_n$.
- Sommers proved the implication (3) $\implies$ (5) for Weyl groups, by considering the quotient Q/kQ of the root lattice Q.
- Berest–Etingof–Ginzburg used the representation theory of rational Cherednik algebras to prove that,
  - if W is a Coxeter group and $k \equiv 1 \bmod h$, or
  - if W is of type B_n, k is odd and $\gcd(k, n) = 1$, or
  - if W is of type D_n, k is odd and $\gcd(k, n-1) = 1$,

  then $\widetilde{\varphi}_k$ is the graded character of a graded representation of W. There are several papers in this direction for some of the exceptional groups.

Idea of Proof

(1) $\widetilde{\varphi}_k$ is the graded character of a graded representation of W.
(2) $\mathrm{Cat}_k^*(W,q)$ is a polynomial in q.
(3) k satisfies a certain congruence condition.
(4) φ_k is the character of a representation of W.
(5) φ_k is the character of a permutation representation of W.

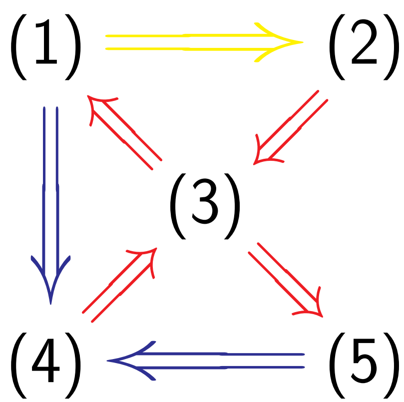


The implications $\Longrightarrow$ are obvious. We can show (1) $\Longrightarrow$ (2) by using

$$\mathrm{Cat}_k^*(W,q) = \text{multiplicity of } \det{}_V \text{ in } \widetilde{\varphi}_k.$$

Idea of Proof

(1) $\widetilde{\varphi}_k$ is the graded character of a graded representation of W.
(2) $\mathrm{Cat}_k^*(W, q)$ is a polynomial in q.
(3) k satisfies a certain congruence condition.
(4) φ_k is the character of a representation of W.
(5) φ_k is the character of a permutation representation of W.



However it is subtle to prove the other implications $\Longrightarrow$. Our proof is case-by-case and uses a computer for the exceptional groups.

Proof for the Symmetric groups
(Greatest Common Divisors of Specialized Schur Functions)

Schur Functions

A partition of a positive integer n is a weakly decreasing sequence

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots), \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

of non-negative integers with $\sum_i \lambda_i = n$. Then we write $\lambda \vdash n$.

Let k be a positive integer. For a partition λ with length $\leq k$, the corresponding Schur function $s_\lambda(x_1, \cdots, x_k)$ is defined by

$$s_\lambda(x_1, \cdots, x_k) = \frac{\det\left(x_i^{\lambda_j+k-j}\right)_{1\leq i,j\leq k}}{\det\left(x_i^{k-j}\right)_{1\leq i,j\leq k}}.$$

We use the following notation for the specialization:

$$s_\lambda(1^k) = s_\lambda(\underbrace{1, \ldots, 1}_{k}).$$

Proof of the Main Theorem for $W = \mathfrak{S}_n$

It follows from the Frobenius formula and the Cauchy formula that

$$\varphi_k = \sum_{\lambda \vdash n} \frac{s_\lambda(1^k)}{k} \chi^\lambda = \sum_{\lambda \vdash n} \frac{m_\lambda(1^k)}{k} \eta^\lambda,$$

where m_λ is the monomial symmetric function and

$\chi^\lambda =$ the irreducible character of $\mathfrak{S}_n$ corresponding to λ,

$\eta^\lambda =$ the permutation character on $\mathfrak{S}_n / \mathfrak{S}_\lambda$.

For the q-analogue $\widetilde{\varphi}_k$, we have

$$\widetilde{\varphi}_k = \sum_{\lambda \vdash n} \frac{s_\lambda(1, q, q^2, \dots, q^{k-1})}{[k]_q} \chi^\lambda.$$

Hence, in order to prove the main theorem for $W = \mathfrak{S}_n$, we need to show the following claims:

- If $s_\lambda(1^k)$ is divisible by k for all $\lambda \vdash n$, then $\gcd(n,k) = 1$.
- If $\gcd(n,k) = 1$, then $m_\lambda(1^k)$ is divisible by k for all $\lambda \vdash n$.
- If $\gcd(n,k) = 1$, then $s_\lambda(1, q, \dots, q^{k-1})$ is divisible by $[k]_q$ for all $\lambda \vdash n$.

These claims are consequences of the following theorems on the greatest common divisors of specialized Schur functions.

Greatest Common Divisors of Specialized Schur Functions

Theorem 1 Let k and n be positive integers. Then we have

$$\gcd{}_{\mathbb{Z}} \left\{ s_\lambda(1^k) : \lambda \vdash n \right\} = \frac{k}{\gcd(n,k)}.$$

Theorem 2 Let k and n be positive integers. Then we have

$$\gcd{}_{\mathbb{Q}[q]} \left\{ s_\lambda(1, q, q^2, \cdots, q^{k-1}) : \lambda \vdash n \right\} = \frac{[k]_q}{[\gcd(n,k)]_q}.$$

Proof of the Main Theorem for $W = \mathfrak{S}_n$

- If $s_\lambda(1^k)$ is divisible by k for all $\lambda \vdash n$, then $\gcd(n,k) = 1$.
- If $\gcd(n,k) = 1$, then $m_\lambda(1^k)$ is divisible by k for all $\lambda \vdash n$.
- If $\gcd(n,k) = 1$, then $s_\lambda(1, q, \ldots, q^{k-1})$ is divisible by $[k]_q$ for all $\lambda \vdash n$.

Greatest Common Divisors of Specialized Schur Functions

Theorem 1 Let k and n be positive integers. Then we have

$$\gcd{}_{\mathbb{Z}}\left\{s_\lambda(1^k) : \lambda \vdash n\right\} = \frac{k}{\gcd(n,k)}.$$

Theorem 2 Let k and n be positive integers. Then we have

$$\gcd{}_{\mathbb{Q}[q]}\left\{s_\lambda(1,q,q^2,\cdots,q^{k-1}) : \lambda \vdash n\right\} = \frac{[k]_q}{[\gcd(n,k)]_q}.$$

Remark Theorem 2 does not imply Theorem 1 by letting $q=1$. For example,

$$\lim_{q\to 1}\gcd\{(q^2+1)(q+1)^2,(q+1)^3\} = \lim_{q\to 1}(q+1)^2 = 4,$$

$$\gcd\{\lim_{q\to 1}(q^2+1)(q+1)^2,\lim_{q\to 1}(q+1)^3\} = \gcd(8,8) = 8.$$

Proof of Theorem 1

Theorem 1 Let k and n be positive integers. Then we have

$$\gcd{}_{\mathbb{Z}}\left\{s_\lambda(1^k) : \lambda \vdash n\right\} = \frac{k}{\gcd(n,k)}.$$

Proof Let $d = \gcd(n,k)$. Theorem 1 follows from the following two claims.

Claim 1 For any partition λ of n, the integer $s_\lambda(1^k)$ is divisible by k/d.

Claim 2 Let p be a prime. If p^L divides all $e_\lambda(1^k)$'s with $\lambda \vdash n$, then p^L divides k/d, where e_λ denotes the elementary symmetric function.

Proof of Theorem 1 (1/4)

Claim 1 For any partition λ of n, the integer $s_\lambda(1^k)$ is divisible by $k/\gcd(n,k)$.

Proof of Claim 1

Let $d = \gcd(n,k)$. It follows from the Frobenius formula that

$$\sum_{\lambda \vdash n} \frac{s_\lambda(1^k)}{k/d} \chi^\lambda(\sigma) = \frac{1}{k/d} \cdot k^{l(\mathrm{type}(\sigma))} \quad (\sigma \in \mathfrak{S}_n),$$

where χ^λ is the irreducible character of the symmetric group $\mathfrak{S}_n$ corresponding to λ, and $\mathrm{type}(\sigma)$ is the cycle type of σ. Hence it is enough to show that there exists a representation of $\mathfrak{S}_n$ whose character θ is given by

$$\theta(\sigma) = \frac{1}{k/d} \cdot k^{l(\mathrm{type}(\sigma))} \quad (\sigma \in \mathfrak{S}_n).$$

Proof of Theorem 1 (2/4)

Consider the permutation representation of $\mathfrak{S}_n$ on $X = (\mathbb{Z}/k\mathbb{Z})^n$, and put

$$X_p = \{(x_i) \in (\mathbb{Z}/k\mathbb{Z})^n : x_1 + \cdots + x_n - pd \in \{0, 1, \cdots, d-1\}\}$$

for $p = 0, 1, \cdots, k/d-1$, where we identify $\mathbb{Z}/k\mathbb{Z}$ with $\{0, 1, \cdots, k-1\}$. If we denote by ψ and ψ_p the permutation character of X and X_p, then we have

$$\psi(\sigma) = k^{l(\mathrm{type}(\sigma))}, \quad \text{and} \quad \psi = \psi_0 + \psi_1 + \cdots + \psi_{k/d-1}.$$

Since $\gcd(k/d, n/d) = 1$, we can find an equivariant bijection between X_0 and X_p, so we have

$$\psi_0 = \psi_1 = \cdots = \psi_{k/d-1}.$$

Hence we conclude that θ is the permutation character ψ_0 of X_0, and that $\frac{s_\lambda(1^k)}{k/d}$ is an integer.

Proof of Theorem 1 (3/4)

Claim 2 Let p be a prime. If p^L divides all $e_\lambda(1^k)$'s with $\lambda \vdash n$, then p^L divides k/d, where e_λ denotes the elementary symmetric function.

Proof of Claim 2 For a positive integer x, we denote by $\nu_p(x)$ the highest power of p dividing x. If we put

$$L = \nu_p\left(\gcd{}_{\mathbb{Z}}\left\{e_\lambda(1^k) : \lambda \vdash n\right\}\right), \quad K = \nu_p(k),$$

we show that $L \leq \nu_p(k/d)$. We write $n = s \cdot p^K + r$ with $0 \leq r < p^K$, and consider the partition $\lambda = (\underbrace{p^K, \dots, p^K}_{s}, r)$ of n. Since we have

$$e_\lambda(1^k) = \prod_i \binom{k}{\lambda_i},$$

we see that

$$L \leq \nu_p\left(e_\lambda(1^k)\right) = \nu_p\left(\binom{k}{r}\right).$$

Proof of Theorem 1 (4/4)

We have

$$L \leq \nu_p\left(e_\lambda(1^k)\right) = \nu_p\left(\binom{k}{r}\right).$$

where $n = s \cdot p^K + r$.

Now we may assume $r > 0$. If we put $R = \nu_p(r)$, then Kummer's theorem tells us that

$$\nu_p\left(\binom{k}{r}\right) = K - R.$$

And it follows from $n = s \cdot p^K + r$ that $\nu_p(n) = R$. Hence we have

$$\nu_p\left(\frac{n}{\gcd(n,k)}\right) = K - R.$$

Theorem 2 Let k and n be positive integers. Then we have

$$\gcd{}_{\mathbb{Q}[q]}\left\{s_\lambda(1, q, q^2, \cdots, q^{k-1}) : \lambda \vdash n\right\} = \frac{[k]_q}{[\gcd(n,k)]_q}.$$

Proof We note that

$$\frac{[k]_q}{[\gcd(n,k)]_q} = \prod_{d|k,\ d\nmid n} \prod_{\zeta} (q - \zeta),$$

where ζ runs over all primitive d-th roots of unity. It suffices to show

1. We have

$$\{q \in \mathbb{C} : q \text{ is a common root of } h_\lambda(1, q, \dots, q^{k-1})\ (\lambda \vdash n)\}$$
$$= \bigsqcup_{d|k,\ d\nmid n} \{q \in \mathbb{C} : z \text{ is a primitive } d\text{-th root of } 1\}.$$

2. If q is a common root of $h_\lambda(1, q, \dots, q^{k-1})$ ($\lambda \vdash n$), then q is a simple root of $h_\mu(1, q, \dots, q^{k-1})$ for some $\mu \vdash n$.

Conjecture

Theorem 2 implies that

$$\frac{s_\lambda(1, q, \cdots, q^{k-1})}{[k]_q/[d]_q} = \frac{s_\lambda(1, q, \cdots, q^{k-1})}{1 + q^d + \cdots + q^{k-d}} \in \mathbb{Z}[q],$$

where $\lambda \vdash n$ and $d = \gcd(n, k)$. In fact we can show

Proposition If λ is a partition of n and $d = \gcd(n, k)$, then

$$\frac{s_\lambda(1, q, \cdots, q^{k-1})}{1 + q^d + \cdots + q^{k-d}} \in \mathbb{N}[q],$$

i.e., it is a polynomial with non-negative integer coefficients.

A finite sequence $(a_0, a_1, \cdots, a_m)$ is called unimodal if there is an index p satisfying

$$a_0 \le a_1 \le \cdots \le a_{p-1} \le a_p \ge a_{p+1} \ge \cdots \ge a_{m-1} \ge a_m.$$

Conjecture Let λ be a partition of n and $d = \gcd(n, k)$. If we write

$$\frac{s_\lambda(1, q, \cdots, q^{k-1})}{1 + q^d + \cdots + q^{k-d}} = \sum_{i \ge 0} a_i q^i,$$

then the sequences

$$(a_0, a_2, a_4, \cdots), \quad \text{and} \quad (a_1, a_3, a_5, \cdots)$$

are both unimodal.

This conjecture is true if

- n is a multiple of k (i.e., $d = k$) (well-known), or
- k is relatively prime to n (consequence of a result of Berest–Etingof–Ginzburg, Gordon).

Problems

1. Give a classification-free proof to the Main Theorem.
2. Give a combinatorial interpretation of
$$\frac{[\gcd(a,b)]_q}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q.$$
3. Prove the following conjecture:

   Conjecture Let λ be a partition of n and $d = \gcd(n,k)$. If we write
$$\frac{s_\lambda(1,q,\cdots,q^{k-1})}{1+q^d+\cdots+q^{k-d}} = \sum_{i\geq 0} a_i q^i,$$
   then the sequences $(a_0, a_2, a_4, \cdots)$ and $(a_1, a_3, a_5, \cdots)$ are both unimodal.