Tree-chromatic number is not equal to path-chromatic number

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Tree-Decomposition and Path-Decomposition

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- A *tree-decomposition* of a graph G is a pair $(T, (X_t : t \in V(T)))$, where T is a tree and $(X_t : t \in V(T))$ is a collection of subsets of vertices of G satisfying
  - $V(G) = \cup_{t \in V(T)} X_t$;
  - for every edge uv of G, there exists $t \in V(T)$ with $u, v \in X_t$;
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- A tree decomposition $(T, (X_t : t \in V(T)))$ is a *path decomposition* if T is a path.

We call X_t a *bag*.

Tree-Chromatic Number and Path-Chromatic Number

The *chromatic number* of $(T, (X_t))$ is $\max_{t \in V(T)} \chi(G[X_t])$.

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The *tree-chromatic number* $\chi_T(G)$ of G is the minimum chromatic number taken over all tree-decompositions of G.

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- $\chi_T(G) \leq \chi_P(G)$

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- (Seymour 2015) no function f such that $\chi_T(G) \leq f(\omega(G))$
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- increases χ_P but preserves ω
  $\Longrightarrow$ graphs with small ω and large χ_P.
- constructs a graph with $\chi_T \neq \chi_P$
  $\Longrightarrow$ gives an answer for the first question.

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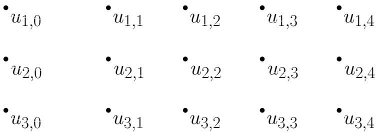

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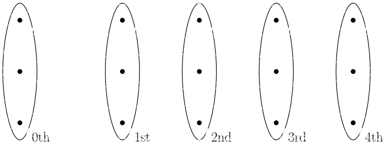

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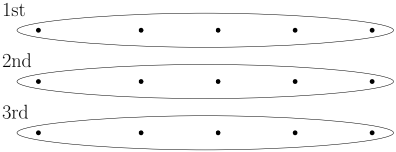
  - i-th row $= \{u_{i,j} \mid 0 \leq j \leq m\}$



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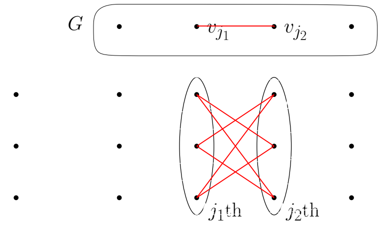

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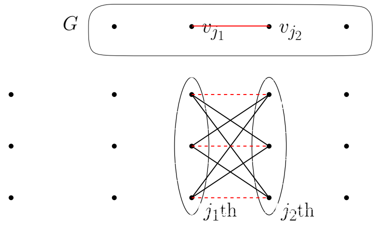

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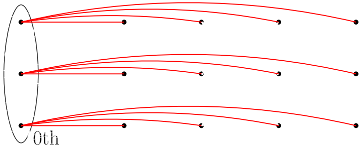

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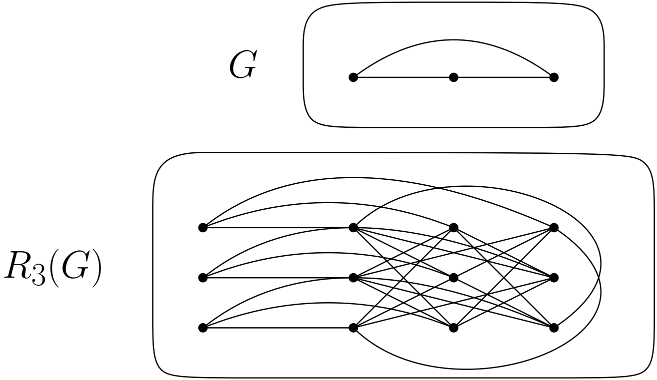

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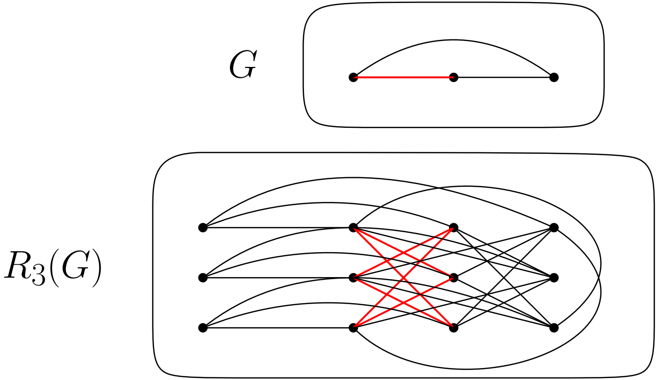

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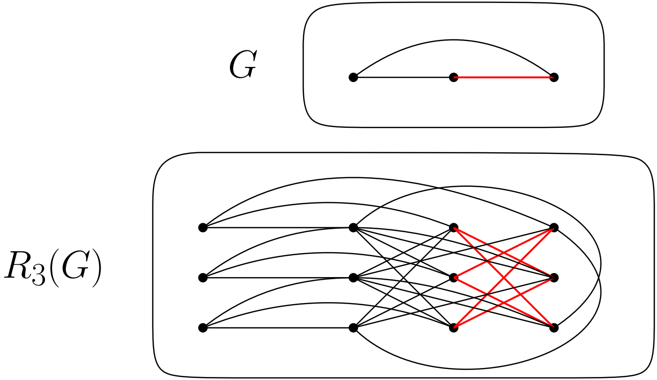

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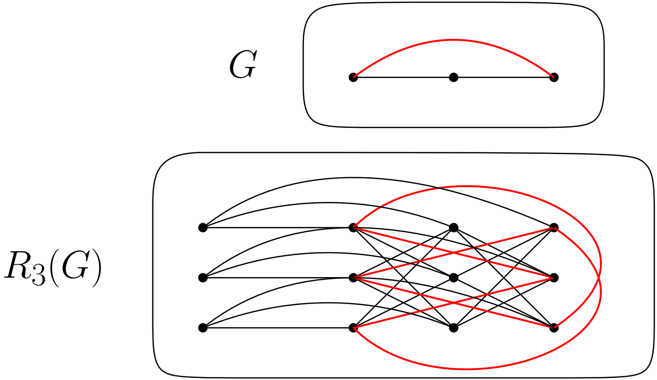

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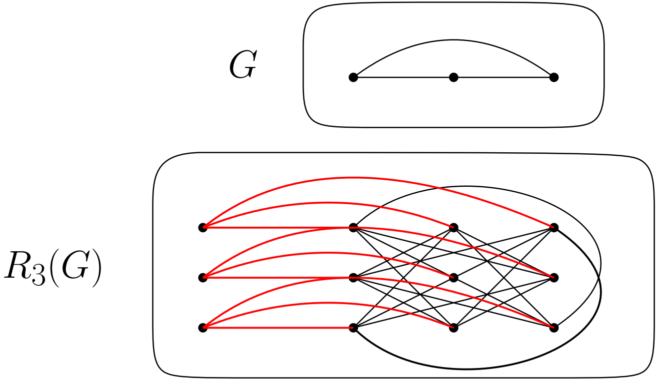

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- $i_1 = i_2$ and either j_1 or j_2 is zero, or
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- each column induces a stable set

Main Theorem

Theorem (T. Huynh, R. K 2015)

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- Taking R_m once does not guarantee an increase of χ_P.

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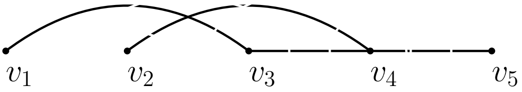

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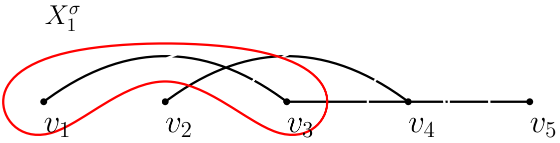

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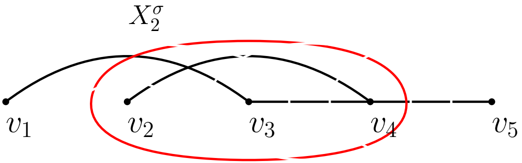

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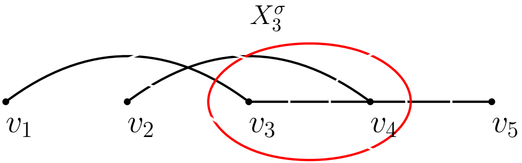

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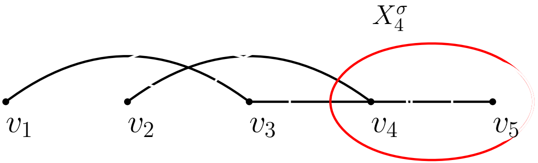

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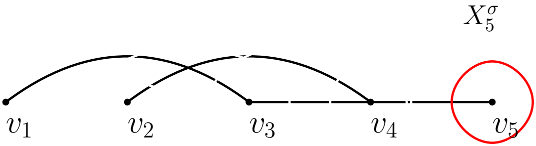

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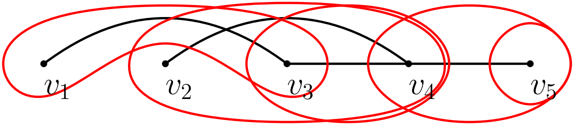
- (P_{n-1}, X^σ) is a path decomposition of G



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$\chi_P(R_m(G))$ is k if there is an enumeration σ of $V(G)$ such that

- (P_{n-1}, X^σ) has path chromatic number k;
- for every $v_i \in V(G)$ with $\chi(G[X_i^\sigma]) = k$, v_i has no neighbors in $\{v_1, v_2, \ldots, v_{i-1}\}$.

Otherwise, the path chromatic number of $R_m(G)$ is $k + 1$.

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Lemma3

Suppose $\chi_P(G) = \chi_P(R_m(G))$. Then, there is no enumeration of $V(R_m(G))$ satisfying conditions in Lemma 2.

Graphs with $\chi_T \neq \chi_P$

Corollary

k: odd integer, m: large integer, C_k: cycle of length k
The path chromatic number of $R_m(C_k)$ is 3 but tree chromatic number is 2.

Proof.

- Evidently, $\chi_P(C_k) = 2$.

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- $R_m(C_k)$ has $\chi_P = 3$.

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- $R_m(C_k)$ has $\chi_P = 3$.
- $\exists (T, X_t)$ of $R_m(C_k)$ with $\chi = 2$. □

Thank you!!