On the distribution of rank-type functions

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Integer Partitions

Definition. A *partition* of a positive integer n is a weakly decreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = n$. The λ_i are called the *parts* of the partition.

Definition. $p(n) =$ the number of partitions of n.
For convenience, we define $p(0) = 1$.

Example. $p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Arithmetic of partition function

Theorem (Ramanujan, 1920)

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}$$

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- A natural question

Can we explain these partition congruences combinatorially?

Dyson's rank

Definition. The rank of partition $\lambda = \lambda_1 - \ell(\lambda)$, where λ_1 is the largest part of λ and $\ell(\lambda)$ is the number of parts of λ.

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- For example,

Partitions of 4
4
3+1
2+2
2+1+1
1+1+1+1

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- For example,

Partitions of 4	rank
4	$4-1=3$
3+1	$3-2=1$
2+2	$2-2=0$
2+1+1	$2-3=-1$
1+1+1+1	$1-4=-3$

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- For example,

Partitions of 4	rank	rank (mod 5)
4	$4-1=3$	3
3+1	$3-2=1$	1
2+2	$2-2=0$	0
2+1+1	$2-3=-1$	4
1+1+1+1	$1-4=-3$	2

Dyson's conjecture

Conjecture (1944)

Let $N(i, p, n)$ be the number of partitions of n with rank $\equiv i \pmod{p}$, then

$$N(i, 5, 5n+4) = \frac{1}{5}p(5n+4),$$
$$N(i, 7, 7n+5) = \frac{1}{7}p(7n+5)$$

for all i.

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for all i.

Theorem (Atkin and Swinnerton-Dyer, 1954)

Dyson's conjecture is true.

Rank generating function

- Let $N(m,n)$ be the number of partitions of n with rank m. Then,

$$R(x,q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) x^m q^n$$
$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-xq^k)(1-q^k/x)}.$$

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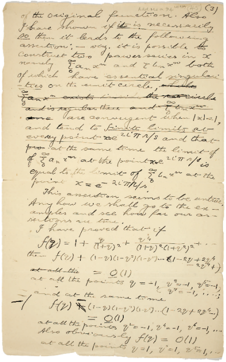
- Ramanujan's mock theta function $f(q)$

$$f(q) = R(-1,q) = \sum_{n=0}^{\infty} (N_e(n) - N_o(n)) q^n$$
$$= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + \cdots,$$

where $N_e(n)$ (resp. $N_o(n)$) is the number of partitions of n with even (resp. odd) rank.

Ramanujan's last letter to Hardy


(3)

of the original function. Also I have shown if it is necessarily so then it leads to the following assertion:— viz. it is possible to construct two power series in x namely $\sum_0^\infty a_n x^n$ and $\sum b_n x^n$ both of which have essential singularities on the unit circle, are convergent when $|x|<1$, and tend to finite limits at every point $x = e^{2i\pi r/s}$ and that at the same time the limit of $\sum_0^\infty a_n x^n$ at the point $x=e^{2i\pi r/s}$ is equal to the limit of $\sum_0^\infty b_n x^n$ at the point $x = e^{-2i\pi r/s}$.

This assertion seems to be untrue. Anyhow we shall go to the examples and see how far our assertions are true.

I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

then $f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)$
$= O(1)$
at all the points $q=-1,\ q^3=-1,\ q^5=-1,\ q^7=-1, \ldots$;
and at the same time
$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots)$
$= O(1)$
at all the points $q^2=-1,\ q^4=-1,\ q^6=-1, \ldots$
Also obviously $f(q) = O(1)$
at all the points $q=1,\ q^3=1,\ q^5=1, \ldots$

"I discovered very interesting functions recently which I call mock θ-functions. Unlike the false θ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions...."

(Ordinary) Theta functions

- Ramanujan's theta function

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |a|,|b| < 1$$

- For example,

$$f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$
$$= \theta(z),$$

where $q = \exp(2\pi i z)$ and $z \in \mathbb{H}$, complex upper half plane.

- Theta functions are modular forms.

Modular Forms

- A modular form of weight k with level N is a function on the complex upper half plane $\mathbb{H}$ such that
  1. f is holomorphic on $\mathbb{H}$.
  2. $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.
  3. f satisfies a certain growth condition.

- Here, $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | c \equiv 0 \pmod{N}\}$ is a congruence subgroup.

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- Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, f has a Fourier expansion

$$f(z) = \sum a(n) q^n, \quad q = \exp(2\pi i z).$$

Merits of Modular Forms

- For fixed weight k and level N, $M_k(\Gamma_0(N))$ is a finite dimensional vector space.

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- Let $r_4(n)$ be the number of representations of n by sums of four squares.

$$\sum_{n\geq 0} r_4(n)q^n = \theta^4(z) = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4$$
$$= 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + \cdots .$$

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- The function

$$E(z) = 1 + \sum_{n\geq 1} 8(\sigma(n) - 4\sigma(n/4))q^n$$
$$= 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + \cdots .$$

Merits of Modular Forms(Cont'd)

• $E(z)$ and $\theta^4(z)$ are modular forms with $k = 2$ and $N = 4$. The dimension of such modular forms is 2, and thus $E(z) = \theta^4(z)$.

• There are many tools on modular form spaces, which encodes arithmetic information of Fourier coefficients. For example,

1. Congruences for the coefficients
2. Growth of coefficients
3. Modular L-functions, and etc.

Mock theta functions

- Ramanujan introduced 17 mock theta functions.
- He listed some asymptotic behaviors.
- He listed some linear relations among them. For example,

$$f(q)+\psi(-q)=\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\right)^2 \prod_{n=1}^{\infty}\frac{1}{1-q^n},$$

where

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\prod_{k=1}^{n}(1+q^k)^2} \quad \text{and} \quad \psi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\prod_{k=1}^{n}(1-q^{2k-1})}$$

are 3rd order mock theta functions.

- A prototype of mock modular forms: S. Zwegers, Bringmann-Ono, Zagier, and many more.

Unimodal sequences

- Unimodal sequences are sequences of the form

$$a_1 \le a_2 \le \cdots \le a_r \le \overline{c} \ge b_1 \ge b_2 \ge \cdots \ge b_s$$

with weight $n = c + \sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i$.

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- Weight 4 unimodal sequences are

$$(\overline{4}), (1, \overline{3}), (\overline{3}, 1), (1, \overline{2}, 1), (\overline{2}, 2), (2, \overline{2}),$$
$$(1, 1, \overline{2}), (\overline{2}, 1, 1), (\overline{1}, 1, 1, 1), (1, \overline{1}, 1, 1), (1, 1, \overline{1}, 1), (1, 1, 1, \overline{1}).$$

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- Weight 4 unimodal sequences are

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- The rank of a unimodal sequence is $s - r$.

Generating function

- Unimodal rank generating function

$$U(x;q) = \sum_{m,n} u(m,n)x^m q^n = \sum_{n\geq 0} \frac{q^n}{\prod_{k=1}^{n}(1-xq^k)(1-q^k/x)}$$

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- Actually, $U(x,q)$ and $R(x,q)$ is connected via replacing q by q^{-1} :

$$U(x, q^{-1}) = R(x;q)$$

In particular,

$$U(-1;q^{-1}) = \sum \frac{q^{n^2}}{\prod_{k=1}^n (1+q^k)^2} = f(q)$$

Generating functions (cont'd)

- Generating function for $u(n)$, the number of unimodal sequences of weight n.

$$\sum_{n=0}^{\infty} u(n)q^n = \sum_{n=0}^{\infty} \frac{q^n}{\prod_{k=1}^{n}(1-q^k)^2}$$
$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

- For a fixed integer m, let $N(m,n)$ be the number of partitions of n with rank m, then

$$\sum_{n\geq 0} N(m,n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n-1)/2+|m|n}(1-q^n).$$

Ramanujan's last letter to Hardy

- Last letter revisited:

"I discovered very interesting functions recently which I call mock θ-functions. Unlike the false θ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions...."

- A partial theta function is a sum of the form

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n-1)/2} x^n.$$

And a false theta function is a theta-like function with wrong sign. For example,

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n-1)/2}(1-q^n).$$

A strange identity of Zagier

- Zagier (2001) proved that

$$F(q) := 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - q^k) \text{ “ = ” } -\frac{1}{2} \sum_{n=1}^{\infty} n\chi_{12}(n) q^{(n^2-1)/24},$$

where $\chi_{12}(12m \pm 1) = 1$ and $\chi_{12}(12m \pm 5) = -1$.

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where $\chi_{12}(12m \pm 1) = 1$ and $\chi_{12}(12m \pm 5) = -1$.

- Euler's Pentagonal number theorem

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{n \geq 1} \chi_{12}(n) q^{(n^2-1)/24},$$

where $q = \exp(2\pi i z)$ and $z \in \mathbb{H}$.

- For $x \in \mathbb{Q}$ and $x \neq 0$, Zagier showed that $\phi(x) = \exp\left(-\frac{\pi i x}{12}\right) F(e^{-2\pi i x})$ transforms similarly to modular forms.

Quantum Modular Forms

- Zagier (2010) introduced Quantum modular forms.

Definition. A quantum modular form of weight k and multiplier system χ on Γ is a function $f(x)$ on $\mathbb{Q}$ such that for every $\gamma \in \Gamma$ the function

$$f(x) - (f|_{k,\chi}\gamma)(x)$$

can be extended smoothly to $\mathbb{R}$ except finitely many points.

- Here, the operator $|_{k,\chi}\gamma$ is defined by

$$(f\,|_{k,\chi}\,\gamma)(x) := \overline{\chi(\gamma)}(cx+d)^{-k} f(\gamma x)\,.$$

- For a modular form $f(z)$, we have $f\,|_{k,\chi}\,(z) = f(z)$.

Unimodal ranks

- Variety types of unimodal sequences according to conditions on the peak and/or the sequences a_n and b_n. For example,
  1. Strong Unimodal sequences (Bryson-Ono-Pitman-Rhoades)
  2. Unimodal sequence with double peak and etc. (K.-Lovejoy)
  3. Odd balanced unimodal sequences and etc. (K.-S. Lim-Lovejoy)

- Connection to Quantum modular forms :
  1. Strong unimodal rank (Bryson-Ono-Pitman-Rhoades) : weight 1/2 quantum modular form
  2. Odd balanced unimodal rank (K.-S. Lim-Lovejoy) : weight 3/2 quantum modular form

Partition Ranks

- Recall the generating function

$$R(x,q) = 1 + \sum_{n\geq 1} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-xq^k)(1-q^k/x)}.$$

- First few coefficients

$$\begin{aligned} R(x,q) &= 1 + q + (x + 1/x)q^2 + \cdots \\ &+ (x^8 + x^6 + x^5 + 2x^4 + 2x^3 + 3x^2 + 3x + 4 \\ &+ 3x^{-1} + 3x^{-2} + 2x^{-3} + 2x^{-4} + x^{-5} + x^{-6} + x^{-8})q^9 + \cdots \end{aligned}$$

- Numerical evidence suggests that for n fixed and large enough, the sequences $\{N(m,n)\}_{m\geq 0}$ is decreasing.

Inequalities on partition ranks

• Song Heng Chan and Renrong Mao (2014) proved that for any nonnegative integers m and n,

$$N(m, n) \geq N(m + 2, n).$$

• Their proof is combinatorial and based on rearrangement of parts.

$$1 + \sum_{n \geq 1} \frac{q^{n^2}}{\prod_{k=1}^{n} (1 - xq^k)(1 - q^k/x)}.$$

Inequalities on partition ranks (Con'td)

- Asymptotically, as $n \to \infty$, for a fixed m,

$$N(m,n) \sim \frac{1}{\sqrt{2\pi}}\left[\frac{1}{4}Y_{5/2}(n) + \frac{17}{96}Y_{7/2}(n) + \left(\frac{1237}{4608} - \frac{m^2}{16}\right)Y_{9/2}(n)\right]$$

where $Y_s = \left(\frac{\pi}{\sqrt{6n}}\right)^s I_{-s}\left(\pi\sqrt{\frac{2n}{3}}\right)$.

- Here, $I_s(z) \sim \frac{e^z}{\sqrt{2\pi z}}$.

- This shows that, for a fixed non-negative integer m, $N(m,n) > N(m+1,n)$ for large enough n.

Unimodal Ranks

- Recall the generating function

$$U(x,q)=\sum_{m,n}u(m,n)x^mq^n=\sum_{n\geq 0}\frac{q^n}{\prod_{k=1}^{n}(1-xq^k)(1-q^k/x)}$$

- First few coefficients

$$\begin{aligned}U(x,q)&=1+q+(x+1+x^{-1})q^2+\cdots+(x^8+x^7+3x^6\\&+4x^5+9x^4+12x^3+20x^2+22x+26+22x^{-1}+20x^{-2}\\&+12x^{-3}+9x^{-4}+4x^{-5}+3x^{-6}+x^{-7}+x^{-8})q^9+\cdots\end{aligned}$$

- Numerical evidence suggests that for n fixed and large enough, the sequences $\{u(m,n)\}_{m\geq 0}$ is decreasing.

Unimodality of Ranks

Theorem (Bringmann-K.)

For $m \in \mathbb{N}_0$, we have, as $n \to \infty$,

$$u(m,n) = \frac{\pi^2}{2} X_3(n) + \frac{\pi^3}{3} X_4(n) + \frac{\pi^4}{72}\left(59 - 36m^2\right) X_5(n) + O\left(n^{-3} e^{2\pi\sqrt{\frac{n}{3}}}\right),$$

where $X_k(n) := (2\sqrt{3n})^{-k} I_{-k}(2\pi\sqrt{n/3})$.

- This implies $u(m,n) > u(m+1,n)$ for large enough n.
- S.-H. Chan and R. Mao's argument should work here, too, i.e. $u(m,n) \geq u(m+2,n)$.

Hardy-Ramanujan Circle method

- Let $q = \exp(2\pi iz)$ with $z \in \mathbb{H}$.
- For $F(q) = \sum_{n\geq 0} a(n)q^n$, Cauchy Theorem implies that

$$a(n) = \frac{1}{2\pi i}\int_C \frac{F(q)}{q^{n+1}}dq.$$

- Integer partition generating function

$$P(q) = \sum_{n\geq 0} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Hardy-Ramanujan Circle method (Cont'd)

Theorem (Hardy-Ramanujan(1918))

As $n \to \infty$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Hardy-Ramanujan Circle method (Cont'd)

Theorem (Hardy-Ramanujan(1918))

As $n \to \infty$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

- They Actually obtained a series expansion.
- Modular transformation implies that

$$P\left(\exp\left(\frac{2\pi i(k+iz)}{h}\right)\right) = A_{h,k} \exp\left(\frac{\pi(z^{-1}-z)}{12k}\right) P\left(\exp\left(\frac{2\pi i(h'+iz^{-1})}{k}\right)\right),$$

where $(h,k)=1$, $h'h \equiv -1 \pmod{k}$, and $A_{h,k}$ is a 24-th root of unity.

Wright's work (1968-1972)

- Generating functions

$$\sum_{n=0}^{\infty} u(n)q^n = \sum_{n=0}^{\infty} \frac{q^n}{\prod_{k=1}^{n}(1-q^k)^2}$$

Wright's work (1968-1972)

- Generating functions

$$\sum_{n=0}^{\infty} u(n)q^n = \sum_{n=0}^{\infty} \frac{q^n}{\prod_{k=1}^{n}(1-q^k)^2}$$

- E. M. Wright obtained asymptotic formula for $u(n)$:

$$u(n) \sim \frac{1}{2^3 3^{3/4}} n^{-5/4} \exp\left(\pi\sqrt{\frac{4n}{3}}\right)$$

via the generating function

$$\sum_{n=0}^{\infty} \frac{q^n}{\prod_{k=1}^{n}(1-q^k)^2} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}$$

Wright's circle method

• Generating functions of the form

(an infinite product) × (a linear comb. of partial theta ftn)

• In particular, the infinite product

$$\prod \frac{1}{1-q^n}$$

have a critical pole at $q = 1$.

• For the infinite product part, the behavior on the circle can be obtained from the modular transformation.

• Thus, the key is understanding partial theta functions near $q = 1$ and away from $q = 1$.

Partition rank asymptotics

- For a fixed integer m, let $N(m, n)$ be the number of partitions of n with rank m, then

$$\sum_{n\geq 0} N(m,n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n-1)/2+|m|n}(1-q^n).$$

- Generating functions of the form

(an infinite product) $\times$ (a linear comb. of partial theta ftn)

- Well fitted into Wright's method.

Another approach

- From mock modularity,

Theorem (J. Dousse-M. Mertens(2015))

Under the certain condition on m,

$$N(m,n) = \frac{\pi}{4\sqrt{6n}} sech^2\left(\frac{\pi m}{2\sqrt{6n}}\right) p(n)\left(1 + O\left(\frac{m^{1/3}}{n^{1/4}}\right)\right).$$

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- For other kind of partition statistics, similar property holds.

Theorem (K.-E. Kim-H. Nam)

As $n \to \infty$,

$$C(m,n) = \frac{\pi}{16}\left(1 - \frac{\pi}{4}\right) \operatorname{sech}^2\left(\frac{\pi m}{2\sqrt{n}}\right) n^{-7/4} e^{\pi\sqrt{n}}\left(1 + O\left(\frac{m^{1/3}}{n^{1/4}}\right)\right)$$

Unimodal rank and partial theta functions

- Generating function

$$\sum_{m,n} u(m,n)x^m q^n = \sum_{n\geq 0} \frac{q^n}{\prod_{k=1}^{n}(1-xq^k)(1-q^k/x)}$$

Unimodal rank and partial theta functions

- Generating function

$$\sum_{m,n} u(m,n)x^m q^n = \sum_{n\geq 0} \frac{q^n}{\prod_{k=1}^{n}(1-xq^k)(1-q^k/x)}$$

- $\sum_n u(m,n)q^n$ is a double sum series, which is not suitable for the circle method.

$$\sum_{n\geq 0} u(m,n)q^n = \delta_{m,0} + \frac{-1}{(q)_\infty^2} \sum_{r,n\geq 0} (-1)^{n+r} q^{n(n+1)/2+r(r+1)/2+2rn+|m|r}(1-q^r).$$

- However, for arithmetic properties, the above type of sum could be very useful.

Hecke-type double sum

• Let $v(n)$ denote the number of odd-balanced unimodal sequences of weight $2n+2$ and let $v(m,n)$ denote the number of such sequences having rank m. Then,

$$\mathcal{V}(x,q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} v(m,n)x^m q^n = \sum_{n\geq 0} \frac{(-xq,-q/x)_n q^n}{(q;q^2)_{n+1}}$$

$$= \frac{x}{1+x}\frac{(-q)_\infty}{(q)_\infty}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)(-1)^n x^r q^{n^2+2n+(2n+1)r+r(r+1)/2}$$

Hecke-type double sum

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• By expressing the exponent by an indefinite quadratic form,

$$q^7\mathcal{V}(1,q^8) \equiv \frac{1}{2} \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})\equiv 7 \pmod 8}} q^{N(\mathfrak{a})} \pmod 2,$$

where $K=\mathbb{Q}(\sqrt{2})$.

For asymptotics

- Another generating function from Ramanujan's lost notebook,

$$\sum_{m,n} u(m,n)x^m q^n = \frac{\sum_{n\geq 0}(-1)^n x^{2n+1} q^{\binom{n+1}{2}}}{\prod_{n=1}^{\infty}(1-xq^n)(1-q^n/x)} + (1-x)\sum_{n\geq 0}(-1)^n x^{3n} q^{n(3n+1)/2}(1-x^2q^{2n+1})$$

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- For a Jacobi form (roughly two variable modular form) $\prod_{n=1}^{\infty}(1-xq^n)(1-q^n/x)$ we know transformation formula.

Key asymptotic

The partial theta function ($d \in \mathbb{Q}^+, k \in \mathbb{N}$)

$$F_{d,k}(z;\tau) := \sum_{n\geq 0} \zeta^{kn+d} q^{(kn+d)^2},$$

where $\zeta = \exp(2\pi iz)$ and $q = \exp(2\pi i\tau)$.

Theorem (Bringmann-K.)

The asymptotic expansion

$$F_{d,k}(z;\tau) = \sum_{\ell\geq 0} \frac{(2k\pi iz)^\ell}{\ell!} \left(\frac{\Gamma\left(\frac{\ell+1}{2}\right)}{2(2\pi)^{\frac{\ell+1}{2}} k^{\ell+1}} (-i\tau)^{-\frac{\ell+1}{2}} - \sum_{j=0}^{N} \frac{(2k^2\pi i)^j}{j!} \frac{B_{2j+\ell+1}\left(\frac{d}{k}\right)}{2j+\ell+1} \tau^j \right) + O\left(|\tau|^{N+1}\right),$$

converges for $|z| < \frac{1}{4k}$.

Weak unimodal case

- Cauchy's integral formula and the symmetry $u(-m,n) = u(m,n)$ imply

$$U_m(q) := \sum u(m,n)q^n = 2\int_0^{\frac{1}{2}} U(\zeta;q)\cos(2\pi mz)dz.$$

- Decomposing the generating function as

$$U(\zeta;q) = G_{u,1}(\zeta;q) + G_{u,2}(\zeta;q),$$

where

$$G_{u,1}(\zeta;q) := \frac{\sum_{n\geq 0}(-1)^n\zeta^{2n+1}q^{\frac{n(n+1)}{2}}}{(\zeta q)_\infty(\zeta^{-1}q)_\infty}$$

- Partial theta series equals to

$$\sum_{n\geq 0}(-1)^n\zeta^{2n+1}q^{\frac{n(n+1)}{2}} = q^{-\frac{1}{8}}\left(F_{1,4}\left(z;\frac{\tau}{8}\right) - F_{3,4}\left(z;\frac{\tau}{8}\right)\right).$$

Weak unimodal case(cont'd)

- For $z \in (0, 1/16)$, the asymptotic expansion

$$G_{u,1}(\zeta;q) = \frac{2iq^{-\frac{1}{24}} e^{\frac{\pi i}{6\tau}} \zeta^{-\frac{1}{2}} (1-\zeta) e^{\frac{\pi i z^2}{\tau}}}{\left(1 - e^{\frac{2\pi i z}{\tau}}\right) e^{-\frac{\pi i z}{\tau}}}$$

$$\times \sum_{\ell \geq 0} \frac{(8\pi i z)^{2\ell}}{(2\ell)!} \sum_{j=0}^{N} \frac{(4\pi i)^j}{j!} \frac{B_{2j+2\ell+1}\left(\frac{1}{4}\right)}{2j+2\ell+1} \tau^j$$

$$+ O\left(|\tau|^{N+1} e^{\frac{83\pi}{768} \mathrm{Im}\left(-\frac{1}{\tau}\right)}\right).$$

- Estimation of $U_m(q)$:

$$U_m(q) = 2\int_0^{\frac{1}{16}} G_{u,1}(\zeta;q) \cos(2\pi m z) dz + \text{error terms}$$

A general asymptotic from Wright's circle method

- Suppose that a function $\mathcal{F}(q) = \sum_{n\geq 0} a(n)q^n$ has the following asymptotic expansion

$$\mathcal{F}(q) = e^{\frac{\pi i}{L\tau}} \sum_{j=1}^{N} A(j)\tau^j + O\left(|\tau|^{N+1} e^{\frac{\pi}{L}\mathrm{Im}\left(-\frac{1}{\tau}\right)}\right),$$

for some $L \in \mathbb{N}, N \in \mathbb{N}$, and $\tau = x + iy$ with $|x| \leq y \to 0$. Moreover, we assume that there exists $\varepsilon > 0$ such that for $y \leq |x| \leq \frac{1}{2}$

$$\mathcal{F}(q) \ll e^{\frac{\pi}{Ly} - \varepsilon}.$$

Theorem (Bringmann-K.)

As $n \to \infty$,

$$a(n) = -2\pi i \sum_{j=1}^{N} A(j) \left(\frac{i}{\sqrt{2Ln}}\right)^{j+1} I_{-j-1}\left(2\pi\sqrt{\frac{2n}{L}}\right) + O\left(n^{-\frac{N+2}{2}} e^{2\pi\sqrt{\frac{2n}{L}}}\right).$$

A Conjecture

It seems that most (actually all cases I know), the coefficient of q^n in rank functions looks unimodal polynomial in x for large enough n.

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It seems that most (actually all cases I know), the coefficient of q^n in rank functions looks unimodal polynomial in x for large enough n.

Suppose that

1. $a(n)$ is increasing. (and coefficients of weakly holomorphic or other interesting forms)
2. $a(m, n)$ is rank function in sense of that $\sum_m a(m, n) = a(n)$.
3. A symmetry $a(m, n) = a(-m, n)$
4. $a(0, n)$ is the max among $a(m, n)$ and $a(m+2, n) < a(m, n)$ (this corresponds to that moving a part before or after the peak, affect the rank 2) holds,

then $a(m, n)$ is unimodal for large enough n.