Polynomials defined by tableaux and linear recurrences

Per Alexandersson, July 2015

Introduction

Famous polynomials:

- *flagged skew Schur polynomials*
- *Hall–Littlewood polynomials*
- *key polynomials, and generalized Demazure atoms*
- *quasi-symmetric Schur polynomials*
- *(dual) Grothendieck polynomials*, etc.

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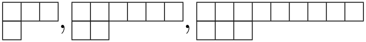
When the underlying fillings (SSYT, SSAF, SVT) satisfy some natural conditions, such polynomials satisfy a linear recurrence.

That is, fix a *shape* α. Then

$$\{P_{k\alpha}(\mathbf{x})\}_{k=1}^{\infty}$$

satisfy a linear recurrence with polynomial coefficients.



Motivation

Let $\mathbf{x} = (x_1, \ldots, x_n)$. Fix λ, n, and consider Schur polynomials:

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}(\lambda, n)} x_1^{w_1(T)} \ldots x_n^{w_n(T)}$$

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Characteristic polynomial of the recurrence:

$$\prod_{T \in \mathrm{SSYT}(\lambda, n)} \left(t - \mathbf{x}^{w(T)}\right).$$

Note: some factors (such as multiple roots) can be removed.

Example, Schur polynomials in two variables

Pick $\lambda = (2,1)$.

$$\mathrm{SSYT}(\lambda, 2) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \right\}.$$

The characteristic polynomial is

$$\prod_{T \in \mathrm{SSYT}(\lambda,2)} \left(t - \mathbf{x}^{w(T)} \right) = t^2 - (x_1^2 x_2 + x_1 x_2^2) t + (x_1^2 x_2 \cdot x_1 x_2^2)$$

so for all $k \geq 1$ one has

$$s_{(k+2)\lambda}(\mathbf{x}) - (x_1^2 x_2 + x_1 x_2^2) s_{(k+1)\lambda}(\mathbf{x}) + (x_1^2 x_2 \cdot x_1 x_2^2) s_{k\lambda}(\mathbf{x}) = 0.$$

Proof that there is a linear recurrence relation

- Look at the ***Weyl character formula***:

$$s_\lambda(x_1,\ldots,x_n) = \underbrace{\frac{1}{\Delta(\mathbf{x})}}_{Vandermonde} \begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \ldots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \ldots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \ldots & x_n^{\lambda_n} \end{vmatrix}$$

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Note that it is straightforward to check for linear recurrences on the computer.

Generalization of the Young tableau method

A *filling* of a diagram is a map $T : D \to \mathbb{N}$, which we represent by writing $T(i,j)$ in the box (i,j).

1	8	2	7	3	4
$\times$	$\times$	9	1		
$\times$	$\times$	5			
$\times$					
$\times$	$\times$	$\times$	9	2	7

Column-closed families of fillings

A family $\mathcal{T}$ of fillings is ***weakly column closed*** if it is closed under duplication of columns:

$$\begin{array}{|c|c|c|c|}\hline 3 & 5 & 2 & \multicolumn{1}{c}{} \\ \hline 2 & 1 & 1 & 1 \\ \hline 3 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \in \mathcal{T} \Longrightarrow \begin{array}{|c|c|c|c|c|}\hline 3 & 5 & 5 & 2 & \multicolumn{1}{c}{} \\ \hline 2 & 1 & 1 & 1 & 1 \\ \hline 3 & 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array} \in \mathcal{T}$$

Example: semi-standard augmented fillings (used for key polynomials and Demazure atoms).

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3	2		

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2	1	1	1	1
3	2	2		

$\in \mathcal{T}$

Example: semi-standard augmented fillings (used for key polynomials and Demazure atoms).

The family is ***strictly column closed*** if we in addition also can remove any column.
Examples: SSYT, RPP, SVT.

Linear statistics

Let $(m_1 C_1, \ldots, m_l C_l)$ to denote the filling with m_1 copies of column C_1 followed by m_2 copies of $C_2 \ldots$.

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Definition

A combinatorial statistic σ is ***affine*** if

$$\sigma(m_1 C_1, \ldots, m_l C_l) = A + S_1 m_1 + S_2 m_2 + \cdots + S_l m_l$$

for all choices of $m_i \geq 1$, where A and S_i are vectors in $\mathbb{N}^s$.

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Note that the statistic given by $w(T) = (w_1, w_2, \ldots, w_n)$, where w_i is the number of boxes filled with i in T, is a linear statistic.

Well-behaved fillings

Definition

A family of fillings is said to be ***well-behaved*** if for every pair of different columns C_1, C_2, there are no fillings T_1, T_2 such that

1. C_1 appears to the left of C_2 in T_1 and
2. C_1 appears to the right of C_2 in T_2.

For example, fillings such that every row is weakly decreasing (or increasing), are well-behaved.

Polynomials

Let D be a fixed diagram and define the polynomial $F_D(\mathbf{z})$ as

$$F_D(\mathbf{z}) = \sum_{T \in \mathcal{T}(D,n)} \mathbf{z}^{\sigma(T)}$$

where $\mathcal{T}(D, n)$ is the set of all fillings $T : D \to [n]$ in $\mathcal{T}$.

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Example (non-symmetric "Hall–Littlewood"):

$$P_\alpha(\mathbf{x}; t) = \sum_{T \in \mathrm{NAWF}(\alpha)} \mathbf{x}^{w(T)} t^{\mathrm{coinv}(T)} (1-t)^{\mathrm{dn}(T)}$$

Polynomials

Theorem (A. '15)

Let $F_D(\mathbf{z})$ be defined on a well-behaved weakly column-closed family via an ***affine*** *statistic. Then the sequence $\{F_{kD}(\mathbf{z})\}_{k=1}^{\infty}$ satisfy a linear recurrence.*

Polynomials

Theorem (A. '15)

Let $F_D(\mathbf{z})$ be defined on a well-behaved weakly column-closed family via an ***affine*** *statistic. Then the sequence $\{F_{kD}(\mathbf{z})\}_{k=1}^{\infty}$ satisfy a linear recurrence.*
Furthermore, if σ is ***linear****, then the characteristic polynomial is given by*

$$\prod_{T} \left(t - \mathbf{z}^{\sigma(T)} \right)$$

where T runs over ***all*** *fillings of shape D such that any adjacent columns of same shape in T are identical, and each T can be obtained from some $\mathcal{T}(kD, n)$ by deletion of some columns.*

Warning: T might *not* itself be an element in $\mathcal{T}$. However, if $\mathcal{T}$ is ***strictly column closed***, then each such T is in $\mathcal{T}(D, n)$.

Proof idea

One column is easy:

3	3		3
5	5	...	5
2	2		2

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$\times$	$\times$		$\times$
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Characteristic polynomial is $\left(t-\mathbf{z}^{\sigma(C_1)}\right)\cdots\left(t-\mathbf{z}^{\sigma(C_l)}\right)$.

Divided difference operators

Let s_i be a simple transposition on indices of variables. Define

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}, \quad \pi_i = \partial_i x_i.$$

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Permutation $\omega \in S_n$ as a reduced word, $\omega = s_{i_1} \cdots s_{i_\ell}$.
Define $\partial_\omega = \partial_{i_1} \cdots \partial_{i_\ell}$ and $\pi_\omega = \pi_{i_1} \cdots \pi_{i_\ell}$.
Then the key polynomials and Schubert polynomials can be computed recursively via

$$\mathcal{K}_\alpha(\mathbf{x}) = \pi_{\omega(\alpha)} x^{\lambda(\alpha)}, \qquad \mathfrak{S}_\omega(\mathbf{x}) = \partial_{(w^{-1}w_0)} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1.$$

Polytopes

A polytope $\mathcal{P}$ is said to have the ***integer decomposition property*** if for every integer $k \geq 1$, every lattice point $\mathbf{x} \in k\mathcal{P}$ can be expressed as $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$ with $\mathbf{x}_i$ lattice points in $\mathcal{P}$.

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$F_1, \ldots, F_\ell$ be faces of a polytope $\mathcal{P}$ with IDP. Define polynomials

$$p_k(\mathbf{z}) = \sum_{\mathbf{x} \in k(\cup_i F_i) \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}}.$$

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Examples: Schur polynomials, skew Schur polynomials, key polynomials.

Conclusion

1. Easy to test for linear recursions.
2. Linear recurrences appear naturally for several different reasons.
3. Might give some hints on the underlying combinatorial statistics.

Thank you for the attention![1]

[1] I am happy to hear about your favorite family of polynomials.