Some congruences involving powers of Legendre polynomials

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1 Introduction

The Legendre polynomials $P_n(x)$ may be defined as

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2} \right)^k .$$

We refer the readers to [1, pp. 1–3]:
Koepf W. Hypergeometric Summation, an Algorithmic Approach to Summation and Special Function Identities. Friedr. Vieweg & Sohn, Braunschweig; 1998.
for seven different definitions of these polynomials. It is well known that the Legendre polynomials have the following orthogonal property:

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{m,n},$$

where $\delta_{m,n}$ denotes the Kronecker delta.

Z.-W. Sun proved that

$$\sum_{k=0}^{p-1}(2k+1)P_k(3) \equiv p + 2p(2^{p-1}-1) - p(2^{p-1}-1)^2 \pmod{p^4},$$

$$\sum_{k=0}^{n-1}(2k+1)P_k^2(3) \equiv 0 \pmod{n^2},$$

where $p > 3$ is a prime.

Z.-W. Sun [6] also raised the following conjecture.

Conjecture 1.1. [6, Conjecture 5.1] *Let x be an integer and let m and n be positive integers. Then*

$$\sum_{k=0}^{n-1}(2k+1)P_k(2x+1)^m \equiv 0 \pmod{n}. \tag{1.1}$$

If p is a prime not dividing $x(x+1)$, then

$$\sum_{k=0}^{p-1}(2k+1)P_k(2x+1)^3 \equiv p\left(\frac{-4x-3}{p}\right) \pmod{p^2}, \tag{1.2}$$

$$\sum_{k=0}^{p-1}(2k+1)P_k(2x+1)^4 \equiv p \pmod{p^2}, \tag{1.3}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

The congruence (1.1) in a more general form has been confirmed by Pan [7]:

Pan H. On divisibility of sums of Apéry polynomials. J Number Theory. 2014;143:214–223.

However, Pan [7] did not give an integer coefficient polynomial formula for

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)P_k(2x+1)^m.$$

We shall prove the following results.

Theorem 1.2. *Let n be a positive integer. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)P_k(2x+1)^3 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{i}\binom{n}{j+k+1}\binom{n+j+k}{j+k}\binom{i+j}{i}\binom{j}{i-k}\binom{j+k}{k}\binom{2i}{i} \times x^{i+j}(x+1)^i, \tag{1.4}$$

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)P_k(2x+1)^4 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{i}\binom{n}{j+k+1}\binom{n+j+k}{j+k}\binom{i+j}{i}\binom{j}{i-k}\binom{j+k}{k}\binom{2i}{i}\binom{2j}{j} \times (x^2+x)^{i+j}. \tag{1.5}$$

Theorem 1.3. *The supercongruences* (1.2) *and* (1.3) *are true.*

Remark. Since $P_n(-x) = (-1)^n P_n(x)$ (see [1, (0.2)]), we see that (1.2) is equivalent to

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)P_k(2x+1)^3 \equiv p\left(\frac{4x+1}{p}\right) \pmod{p^2}. \tag{1.6}$$

For any positive integer n and p-adic integer x, Z.-W. Sun [5, (4.6)] conjectured that

$$\nu_p\left(\frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2k+1)P_k(2x+1)^3\right) \geqslant \min\{\nu_p(n), \nu_p(4x+1)\}. \tag{1.7}$$

where $\nu_p(x)$ denotes the p-adic valuation of x. It is clear that the congruence (1.6) confirms the $n = p$ case of (1.7).

Theorem 1.4. *Let x be an integer and p an odd prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)P_k(2x+1)^4 \equiv p\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{2k}{k}^2(x^2+x)^k(2x+1)^{2k} \pmod{p^2}. \tag{1.8}$$

Numerical calculation suggests the following stronger result.

Conjecture 1.5. *The congruence (1.8) holds modulo p^3.*

2 Proof of Theorem 1.2

It is easy to see that (see [8, Lemma 3.2])

$$P_n(2x+1)^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\binom{2k}{k} x^k (x+1)^k, \qquad (2.1)$$

which can be deduced from Clausen's formula [9] (with $a = -\frac{n}{2}$, $b = \frac{n+1}{2}$ and $x \to -4x(x+1)$):

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix}; x\right]^2 = {}_3F_2\left[\begin{matrix} 2a,\ 2b,\ a+b \\ 2a+2b,\ a+b+\frac{1}{2} \end{matrix}; x\right], \quad |x|<1, \qquad (2.2)$$

and the following quadratic transformation of Gauss hypergeometric function (see [1, p. 180]):

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix}; 4x(1-x)\right] = {}_2F_1\left[\begin{matrix} 2a,\ 2b \\ a+b+\frac{1}{2} \end{matrix}; x\right]. \qquad (2.3)$$

Writing $P_\ell(2x+1)^3$ as $P_\ell(2x+1)^2 \cdot P_\ell(2x+1)$ and applying (2.1),

$$\frac{1}{n}\sum_{\ell=0}^{n-1}(2\ell+1)P_\ell(2x+1)^3 = \frac{1}{n}\sum_{\ell=0}^{n-1}(2\ell+1)\sum_{i=0}^{\ell}\binom{\ell}{i}\binom{\ell+i}{i}\binom{2i}{i}x^i(x+1)^i\sum_{j=0}^{\ell}\binom{\ell}{j}\binom{\ell+j}{j}x^j. \tag{2.4}$$

Note that (see the proof of [10, Lemma 4.2])

$$\binom{\ell}{i}\binom{\ell+i}{i}\binom{\ell}{j}\binom{\ell+j}{j} = \sum_{k=0}^{i}\binom{i+j}{i}\binom{j}{i-k}\binom{j+k}{k}\binom{\ell}{j+k}\binom{\ell+j+k}{j+k}. \tag{2.5}$$

Moreover, we can easily prove that

$$\sum_{\ell=k}^{n-1}(2\ell+1)\binom{\ell}{k}\binom{\ell+k}{k} = n\binom{n}{k+1}\binom{n+k}{k}. \tag{2.6}$$

Substituting (2.5) into (2.4), and utilizing (2.6), we complete the proof of (1.4).
Similarly, writing $P_\ell(2x+1)^4$ as $P_\ell(2x+1)^2 \cdot P_\ell(2x+1)^2$ and applying (2.1), we can prove (1.5).

3 Proof of Theorem 1.3

Proof of (1.2). Letting $n = p$ be a prime in (1.4), and noticing that $\binom{p}{k} \equiv 0 \pmod p$ for $1 \leqslant k \leqslant p-1$ and $\binom{2p-1}{p} \equiv 1 \pmod p$, we obtain

$$\begin{aligned}
&\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)P_k(2x+1)^3 \\
&= \sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\sum_{k=0}^{i}\binom{p}{j+k+1}\binom{p+j+k}{j+k}\binom{i+j}{i}\binom{j}{i-k}\binom{j+k}{k}\binom{2i}{i}x^{i+j}(x+1)^i \\
&\equiv \sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\binom{i+j}{i}\binom{j}{p-i-1}\binom{p-1}{j}\binom{2i}{i}x^{i+j}(x+1)^i \pmod p.
\end{aligned} \tag{3.1}$$

For $0 \leqslant i, j \leqslant p-1$, there holds

$$\binom{i+j}{i}\binom{j}{p-i-1}\begin{cases} = 0, & \text{if } i+j < p-1, \\ \equiv 0 \pmod p, & \text{if } i+j \geqslant p. \end{cases} \tag{3.2}$$

Therefore, the possible nonzero summands modulo p in (3.1) must satisfy $i+j=p-1$. In other words, the congruence (3.1) may be simplified as

$$\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)P_k(2x+1)^3 \equiv \sum_{i=0}^{p-1}\binom{p-1}{i}\binom{p-1}{p-i-1}\binom{2i}{i}x^{p-1}(x+1)^i$$
$$\equiv \sum_{i=0}^{p-1}\binom{2i}{i}(x+1)^i \pmod{p},$$

where we used the fact $\binom{p-1}{i} \equiv (-1)^i \pmod{p}$ and Fermat's little theorem. The proof then follows from the well-known congruence

$$\sum_{k=0}^{p-1}\binom{2k}{k}x^k \equiv \left(\frac{1-4x}{p}\right) \pmod{p} \tag{3.3}$$

(see [6, Lemma 2.1] or [11, Theorem 1.1]). □

Proof of (1.3). Let $n = p$ be a prime in (1.5). Similarly to the proof of (1.2), we have

$$\begin{aligned}
&\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)P_k(2x+1)^4\\
&\equiv \sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\binom{i+j}{i}\binom{j}{p-i-1}\binom{p-1}{j}\binom{2i}{i}\binom{2j}{j}(x^2+x)^{i+j}\\
&\equiv \sum_{i=0}^{p-1}\binom{2i}{i}\binom{2p-2i-2}{p-i-1} \quad \text{(by (3.2) and Fermat's little theorem)}\\
&\equiv 1 \pmod p,
\end{aligned}$$

where in the last step we used the following fact

$$\binom{2i}{i} \equiv 0 \pmod p \quad \text{for} \quad \frac{p-1}{2} < i < p, \tag{3.4}$$

and $\binom{p-1}{\frac{p-1}{2}}^2 \equiv 1 \pmod p$. □

4 Proof of Theorem 1.4

We need the following two lemmas.

Lemma 4.1. *Let n be a positive integer. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1}(-1)^{n-k-1}(2k+1)P_k(2x+1)^4$$
$$=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{i}\binom{n-1}{j+k}\binom{n+j+k}{j+k}\binom{i+j}{i}\binom{j}{i-k}\binom{j+k}{k}\binom{2i}{i}\binom{2j}{j}(x^2+x)^{i+j} \tag{4.1}$$

Proof. It is exactly similar to the proof of Theorem 1.2. The difference is that we need to replace (2.6) by the following identity:

$$\sum_{\ell=k}^{n-1}(-1)^{n-\ell-1}(2\ell+1)\binom{\ell}{k}\binom{\ell+k}{k}=n\binom{n-1}{k}\binom{n+k}{k},$$

which can be proved by induction on n. □

Lemma 4.2. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k} (-4)^{n-k}. \qquad (4.2)$$

Proof. Applying Zeilberger's algorithm (see [1, 12]), we find that both sides of (4.2) satisfy the following recurrence relation:

$$(n+2)^2 S(n+2) - 4(3n^2+9n+7) S(n+1) + 32(n+1)^2 S(n) = 0.$$

Noticing that they also have the same initial values $S(0) = 1$ and $S(1) = 4$, we complete the proof. □

Remark. The referee pointed out that the identity (4.2) is well known: It appeared at least more than 20 years ago, when Apéry's work and its links with hypergeometric functions were discussed. Recently, Z.-W. Sun [13, pp. 11 and 24] gave the following identities:

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k} = \sum_{k=0}^{n}\binom{n}{2k}\binom{2k}{k}^2 4^{n-2k}$$
$$= \sum_{k=0}^{n}\binom{2k}{k}^2\binom{k}{n-k}(-4)^{n-k}.$$

It is worth mentioning that the numbers $2^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{k}{n-k}(-4)^{n-k}$ are called the Catalan-Larcombe-French numbers, of which some supercongruences were studied by Jarvis and Verrill [14].

Proof of (1.8). Let $n = p$ be a prime in (4.1). Then

$$\frac{1}{p}\sum_{k=0}^{p-1}(-1)^k(2k+1)P_k(2x+1)^4 \tag{4.3}$$

$$= \sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\sum_{k=0}^{i}\binom{p-1}{j+k}\binom{p+j+k}{j+k}\binom{i+j}{i}\binom{j}{i-k}\binom{j+k}{k}\binom{2i}{i}\binom{2j}{j}(x^2+x)^{i+j}$$

$$\equiv \sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\sum_{k=0}^{i}(-1)^{j+k}\binom{i+j}{i}\binom{j}{i-k}\binom{j+k}{k}\binom{2i}{i}\binom{2j}{j}(x^2+x)^{i+j} \pmod{p}, \tag{4.4}$$

where we used the fact that, for $0 \leqslant j, k \leqslant p-1$,

$$\binom{p-1}{j+k}\binom{p+j+k}{j+k}\binom{j+k}{k} \equiv (-1)^{j+k}\binom{j+k}{k} \pmod{p}.$$

By the Chu-Vandermonde summation formula, we get

$$\sum_{k=0}^{i}(-1)^k\binom{j}{i-k}\binom{j+k}{k}=(-1)^i. \tag{4.5}$$

Substituting (4.5) into (4.4), we obtain

$$\begin{aligned}
&\frac{1}{p}\sum_{k=0}^{p-1}(-1)^k(2k+1)P_k(2x+1)^4\\
&\equiv\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}(-1)^{i+j}\binom{i+j}{i}\binom{2i}{i}\binom{2j}{j}(x^2+x)^{i+j}\\
&=\sum_{n=0}^{p-1}(-1)^n(x^2+x)^n\sum_{i=0}^{n}\binom{n}{i}\binom{2i}{i}\binom{2n-2i}{n-i} \pmod p,
\end{aligned} \tag{4.6}$$

where we used the congruence $\binom{i+j}{i}\equiv 0 \pmod p$ for $0\leqslant i,j\leqslant p-1$ and $i+j\geqslant p$.

By (4.2), the right-hand side of (4.6) is equal to

$$\begin{aligned}
&\sum_{n=0}^{p-1}(-1)^n(x^2+x)^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{k}{n-k}(-4)^{n-k}\\
&=\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2\sum_{n=k}^{p-1}\binom{k}{n-k}4^{n-k}(x^2+x)^n\\
&\equiv\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{2k}{k}^2\sum_{n=k}^{2k}\binom{k}{n-k}4^{n-k}(x^2+x)^n\\
&=\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{2k}{k}^2(x^2+x)^k(1+4x+4x^2)^k \pmod{p},
\end{aligned}$$

where we used the congruence (3.4) and the binomial theorem. This completes the proof. □

There are several difficult conjectures in the following paper:

V.J.W. Guo and Jiang Zeng, Some q-analogues of supercongruences of Rodriguez-Villegas, J. Number Theory 145 (2014), 301-316.

One of them was just proved:

arXiv:1507.03107 The Rodriguez-Villegas type congruences for truncated q-hypergeometric functions, Victor J. W. Guo, Hao Pan, Yong Zhang. math.NT (math.CO).

We also have some conjectures in

arXiv:1412.5415 Proof of some conjectures of Z.-W. Sun on the divisibility of certain double-sums, Victor J. W. Guo, Ji-Cai Liu. math.NT (math.CO). to appear in Int. J. Number Theory.

Let

$$S_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1)^r, \quad \text{and} \quad T_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1)^r (-1)^k.$$

Numerical calculation suggests the following conjecture.

Conjecture 4.3. *Let n and r be positive integers and p a prime. Then*

$$\sum_{k=0}^{n-1} S_k^{(2r)} \equiv 0 \pmod{n^2}, \tag{4.7}$$

$$\sum_{k=0}^{n-1} T_k^{(2r)} \equiv 0 \pmod{n^2}, \tag{4.8}$$

The congruence (4.7) for $r = 1$ was proved by Z.-W. Sun:
Z.-W. Sun, Two new kinds of numbers and related divisibility results, preprint, 2014, arXiv:1408.5381v8.
The congruence (4.8) for $r = 1$ was proved by Z.-W. Sun and his student. (But they have not yet written the paper!)

Thank you!

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