# Proof of Gessel's $\gamma$-positivity conjecture

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# Eulerian polynomials

## Definition

The Eulerian polynomial $A_n(t)$ may be defined by Euler's basic formula (Leonhard Euler 1755):

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k\geq 1} k^n t^k.$$

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$$\begin{aligned} A_1(t) &= t \\ A_2(t) &= t(1+t) \\ A_3(t) &= t(1+4t+t^2) \\ A_4(t) &= t(1+11t+11t^2+t^3) \\ A_5(t) &= t(1+26t+66t^2+26t^3+t^4) \end{aligned}$$

# Number of descents

$\mathfrak{S}_n$: Set of all permutations of $[n] := \{1, 2, \cdots, n\}$

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For $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, the number of descents of $\pi$:

$$\mathrm{des}(\pi) := \#\{i \in [n-1] : \pi_i > \pi_{i+1}\}.$$

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This is the first $\pi$ with $\mathrm{des}(\pi) \neq \mathrm{ides}(\pi)$. Permutations that avoid both the patterns 2413 and 3142 are separable permutations.

# As descent polynomial

$$A_2(t) = t(1+t) = \sum_{\pi \in \mathfrak{S}_2} t^{\mathrm{des}(\pi)+1}$$

$$A_3(t) = t(1+4t+t^2) = \sum_{\pi \in \mathfrak{S}_3} t^{\mathrm{des}(\pi)+1}$$

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**Theorem (Riordan 1958)**

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)+1}.$$

# A recurrence relation for $A_n(t)$

$$\begin{aligned}
\frac{A_{n+1}(t)}{(1-t)^{n+2}} &= \sum_{k\geq 1} k^{n+1} t^k \\
&= \sum_{k\geq 1}(k-1)k^n t^k + \sum_{k\geq 1} k^n t^k \\
&= t^2\left(\frac{A_n(t)}{t(1-t)^{n+1}}\right)' + \frac{A_n(t)}{(1-t)^{n+1}}
\end{aligned}$$

**Lemma**

*A recurrence relation for Eulerian polynomials:*

$$A_{n+1}(t) = t(1-t)A_n'(t) + (n+1)tA_n(t).$$

# $\gamma$-positivity

A polynomial $h(t) = \sum_{i=0}^{d} h_i t^i$ is

- symmetric if $h_i = h_{d-i}$ for all $i$;
- unimodal if there exists a $m$

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A symmetric polynomial $h(t)$ can be expanded uniquely as

$$h(t) = \sum_{k=0}^{\lfloor d/2 \rfloor} \gamma_k t^k (1+t)^{d-2k}.$$

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$\gamma$-positivity $\Longrightarrow$ symmetry and unimodality (why?)

$\mathrm{NDD}_n$: set of all permutations in $\mathfrak{S}_n$ without double descents

**Theorem (Foata & Schützenberger 1970)**

$$A_n(t) = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \gamma_{n,i} t^i (1+t)^{n+1-2i},$$

*where* $\gamma_{n,i} = \#\{\pi \in \mathrm{NDD}_n : \mathrm{des}(\pi) = i-1, \pi_1 < \pi_2\}$.

# The Eulerian polynomials are $\gamma$-positive

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Many proofs are known: recurrence, Foata-Strehl action, *cd*-index, continued fractions, poset topology (Rees products), ...

# A recurrence relation for $\gamma_{n,k}$

Let $\Gamma_n(X) := \sum_{i\geq 1} \gamma_{n,i} X^i$. Then,

$$A_n(t) = t^{n+1}\Gamma_n(X) \quad \text{with} \quad X = \frac{t}{(1+t)^2}.$$

**Lemma**

*We have*

$$\Gamma_{n+1}(X) = 2(n+1)X\Gamma_n(X) + X(1-4X)\Gamma_n'(X).$$

*Equivalently,*

$$\gamma_{n+1,i} = i\gamma_{n,i} + 2(n+3-2i)\gamma_{n,i-1}.$$

# Foata-Strehl action on permutations

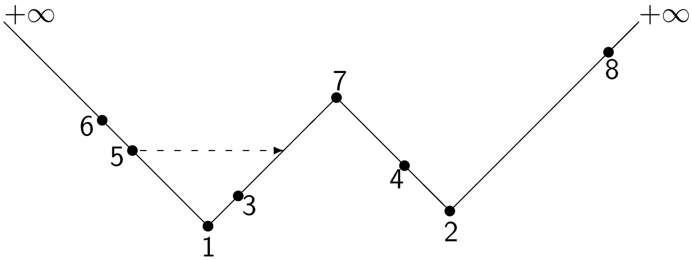


Figure : The Foata–Strehl action $\varphi_x$, where $x = 5$.

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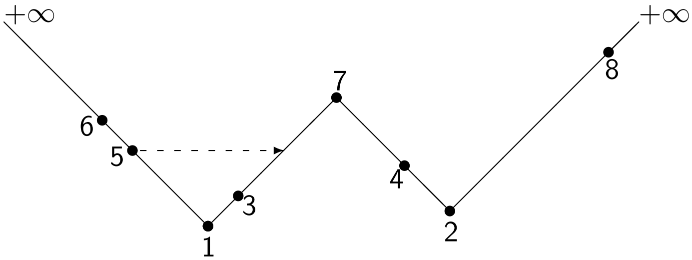


Figure : The Foata–Strehl action $\varphi_x$, where $x = 5$.

A combinatorial proof of:

$$\gamma_{n+1,i} = i\gamma_{n,i} + 2(n+3-2i)\gamma_{n,i-1},$$

where $\gamma_{n,i} = \#\{\pi \in \mathrm{NDD}_n : \mathrm{des}(\pi) = i-1, \pi_1 < \pi_2\}$.

Double Eulerian polynomials (Carlitz-Roselle-Scoville 1966):

$$A_n(s,t) := \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{des}(\pi^{-1})+1} t^{\mathrm{des}(\pi)+1}.$$

**Conjecture (Gessel 2005)**

*The integers $\gamma_{n,i,j}$ are nonnegative in:*

$$A_n(s,t) = \sum_{\substack{i\geq 1, j\geq 0\\ j+2i\leq n+1}} \gamma_{n,i,j}(st)^i(1+st)^j(s+t)^{n+1-j-2i}.$$

# Generalizations

Fix a positive integer $k$. Define the generalized double Eulerian polynomial $A_n^{(k)}(s,t)$ by the identity

$$\frac{A_n^{(k)}(s,t)}{(1-s)^{n+1}(1-t)^{n+1}} = \sum_{i,j\geq 0} \binom{ij+n-k}{n} s^i t^j.$$

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**Theorem (Gessel)**

*For any fixed $\sigma \in \mathfrak{S}_n$ with $\mathrm{des}(\sigma) = k-1$:*

$$A_n^{(k)}(s,t) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{des}(\pi^{-1})+1} t^{\mathrm{des}(\pi\sigma)+1}.$$

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$$A_n^{(k)}(s,t) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{des}(\pi^{-1})+1} t^{\mathrm{des}(\pi\sigma)+1}.$$

- If $\sigma = 1\,2 \cdots n$, then $A_n^{(1)}(s,t) = A_n(s,t)$;
- if $\sigma = n\,(n-1) \cdots 1$, then $A_n^{(n)}(s,t) = t^{n+1} A_n(s, 1/t)$, since $\mathrm{des}(\pi\sigma) = n-1-\mathrm{des}(\pi)$.

Conjectured by Gessel (2005):

**Theorem (L. 2015)**

*For $n \geq 1$ and $1 \leq k \leq n$, we have*

$$A_n^{(k)}(s,t) = \sum_{\substack{i\geq 1, j\geq 0\\ j+2i\leq n+1}} \gamma_{n,i,j}^{(k)} (st)^i (1+st)^j (s+t)^{n+1-j-2i},$$

*where $\gamma_{n,i,j}^{(k)}$ are nonnegative integers.*

# Algebraic proof of $\gamma_{n,i,j} \geq 0$

**Lemma (Petersen 2013)**

$$\begin{aligned} nA_n(s,t) = &(n^2st + (n-1)(1-s)(1-t))A_{n-1}(s,t) \\ &+ nst(1-s)\frac{\partial}{\partial s}A_{n-1}(s,t) + nst(1-t)\frac{\partial}{\partial t}A_{n-1}(s,t) \\ &+ st(1-s)(1-t)\frac{\partial^2}{\partial s \partial t}A_{n-1}(s,t). \end{aligned}$$

Let $\Gamma_n(X,Y) := \sum_{i,j} \gamma_{n,i,j} X^i Y^j$. Observe that Gessel's refined $\gamma$-decomposition is equivalent to the following relationship:

$$A_n(s,t) = (s+t)^{n+1}\Gamma_n(X,Y)$$

with

$$X = \frac{st}{(s+t)^2} \quad \text{and} \quad Y = \frac{1+st}{s+t}.$$

# Algebraic proof of $\gamma_{n,i,j} \geq 0$

**Lemma (Visontai 2013)**

*Let $n \geq 1$. For all $i \geq 1$ and $j \geq 0$, we have*

$$\begin{aligned}
&(n+1)\gamma_{n+1,i,j} \\
&= (n + i(n+2-i-j))\gamma_{n,i,j-1} + (i(i+j)-n)\gamma_{n,i,j} \\
&\quad + (n+4-2i-j)(n+3-2i-j)\gamma_{n,i-1,j-1} \\
&\quad + (n+2i+j)(n+3-2i-j)\gamma_{n,i-1,j} \\
&\quad + (j+1)(2n+2-j)\gamma_{n,i-1,j+1} + (j+1)(j+2)\gamma_{n,i-1,j+2},
\end{aligned}$$

*where $\gamma_{1,1,0} = 1$, $\gamma_{1,i,j} = 0$ (unless $i = 1$ and $j = 0$) and $\gamma_{n,i,j} = 0$ if $i < 1$ or $j < 0$.*

- First derived by Visontai using Eulerian operators and a homogenized multivariate refinement for $A_n(s, t)$;
- here we provide a new direct approach.

# Algebraic proof of $\gamma_{n,i,j} \geq 0$

Key observation (inspired by Computer Data, which can be proved by induction on $n$, carefully!):

**Theorem**

*For $n \geq 1$, the coefficients $\gamma_{n,i,j}$ are nonnegative. Moreover, the coefficient $\gamma_{n,i,j}$ is positive if and only if*

$$i \geq 1, j \geq 0, 2i + j \leq n + 1 \quad \textit{and} \quad i(i + j) \geq n.$$

This result characterizes completely when the coefficient $\gamma_{n,i,j}$ is positive.

**Lemma**

*The generalized double Eulerian polynomial $A_n^{(k)}(s,t)$ satisfy the recurrence relation*

$$\begin{aligned} nA_n^{(k)}(s,t) = {} & (n^2st + (n-k)(1-s)(1-t))A_{n-1}^{(k)}(s,t) \\ & + nst(1-s)\frac{\partial}{\partial s}A_{n-1}^{(k)}(s,t) + nst(1-t)\frac{\partial}{\partial t}A_{n-1}^{(k)}(s,t) \\ & + st(1-s)(1-t)\frac{\partial^2}{\partial s \partial t}A_{n-1}^{(k)}(s,t), \end{aligned}$$

*where*

$$A_k^{(k)}(s,t) = \sum_{\pi \in \mathfrak{S}_k} s^{\mathrm{des}(\pi^{-1})+1} t^{k-\mathrm{des}(\pi)} = t^{k+1} A_k(s, 1/t).$$

**Lemma**

*Fix a positive integer $k$. Let $n \geq k$. Then, for all $i \geq 1$ and $j \geq 0$*

$$\begin{aligned}
&(n+1)\gamma^{(k)}_{n+1,i,j} \\
&= (n+1-k+i(n+2-i-j))\gamma^{(k)}_{n,i,j-1} \\
&\quad + (i(i+j)-(n+1-k))\gamma^{(k)}_{n,i,j} \\
&\quad + (n+4-2i-j)(n+3-2i-j)\gamma^{(k)}_{n,i-1,j-1} \\
&\quad + (n+2i+j)(n+3-2i-j)\gamma^{(k)}_{n,i-1,j} \\
&\quad + (j+1)(2n+2-j)\gamma^{(k)}_{n,i-1,j+1} + (j+1)(j+2)\gamma^{(k)}_{n,i-1,j+2},
\end{aligned}$$

*where* $\gamma^{(k)}_{k,i,j} = \gamma_{k,i,k+1-2i-j}$.

Sum up for all $j$: $\gamma_{n+1,i} = i\gamma_{n,i} + 2(n+3-2i)\gamma_{n,i-1}$.

# Algebraic proof of generalizations

The nonnegativity of $\gamma_{n,i,j}^{(k)}$ is confirmed by the following lemma (by induction on $n$ based on $\gamma_{k,i,j}^{(k)} = \gamma_{k,i,k+1-2i-j} \geq 0$).

**Lemma**

*Fix a positive integer $k$. Let $n \geq k$. Then, for $i \geq 1, j \geq 0$ and $2i + j \leq n + 1$:*

(i) *all the coefficients $\gamma_{n,i,j}^{(k)}$ are nonnegative;*

(ii) *the coefficient $\gamma_{n,i,j}^{(k)}$ vanishes if $i(i+j) < n + 1 - k$.*

# Further work

## Problem

*Is there any combinatorial interpretation for these refined $\gamma$-coefficients $\gamma_{n,i,j}$ (more generally, for $\gamma^{(k)}_{n,i,j}$)?*

## Conjecture (Guo-Zeng 2006)

*Let $\mathcal{I}_n$ be the set of involutions in $\mathfrak{S}_n$. Then,*

$$\sum_{\pi\in\mathcal{I}_n} t^{\mathrm{des}(\pi)} = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} a_{n,i} t^i (1+t)^{n-1-2i},$$

*where $a_{n,i}\in\mathbb{N}$ for $0\le i\le\lfloor\frac{n-1}{2}\rfloor$.*

Merci pour votre attention!