Two New Unimodal Descent Polynomials

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Joint Work with Zhicong Lin and Jiang Zeng

Outline

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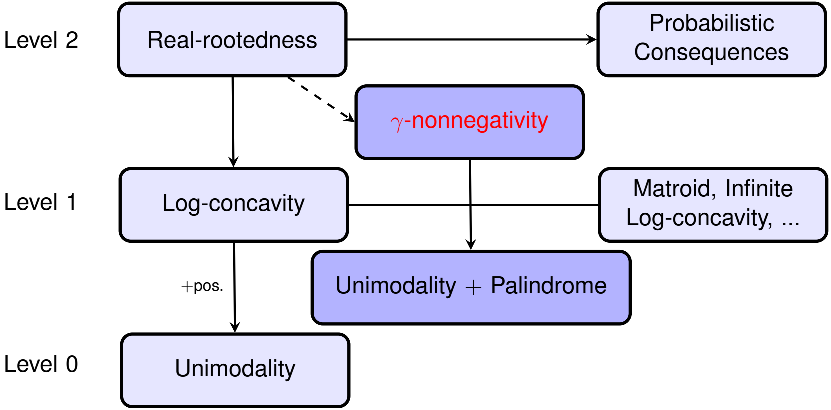
Example

The nth row in Pascal's triangle $\{\binom{n}{k}\}_{k=0}^{n}$, i.e. $(1+x)^n$ satisfies all five properties above.

Hierarchy and More



In-between?

- Log-concave but not real-rooted. The q-factorial:

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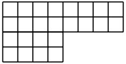
- Not unimodal II. Stanton in 1990. 

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- Classical example includes Foata-Strehl’s “valley-hopping” for Eulerian polynomial.

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- This leads us naturally to the operations $\oplus$ and $\ominus$, and I am about to tell the rest of the story...

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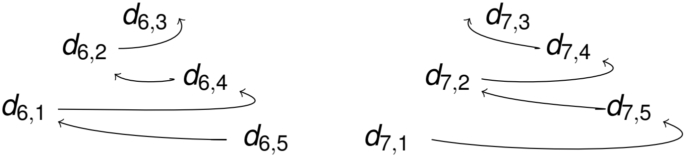

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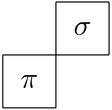
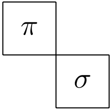
$\pi \oplus \sigma =$ 

$\pi \ominus \sigma =$ 

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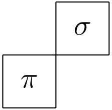
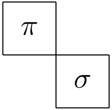
$\pi \oplus \sigma =$ 

$\pi \ominus \sigma =$ 

$$\mathrm{des}(\pi \oplus \sigma) = \mathrm{des}(\pi) + \mathrm{des}(\sigma), \mathrm{ides}(\pi \oplus \sigma) = \mathrm{ides}(\pi) + \mathrm{ides}(\sigma);$$
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ii) We repeat this process until no new blocks can be formed.

iii) The process will end with a single block if and only if it avoids patterns (3142) and (2413). We call such π a *separable permutation*, otherwise non-separable. For separable π, we replace all the numbers with 1 (since their order has been coded by $\oplus$, $\ominus$ and parenthesis) and call the final expression, denoted by $\mathrm{sw}(\pi)$, a *Schröder word*.

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sweep 5: $(9 \ominus 8)\big(4 \ominus \big(1 \oplus (3 \ominus 2)\big)\big)7(5 \oplus 6)$

sweep 6: $(9 \ominus 8)\big(4 \ominus \big(1 \oplus (3 \ominus 2)\big)\big)\big(7 \ominus (5 \oplus 6)\big)$

sweep 7: $(9 \ominus 8)\big(\big(4 \ominus \big(1 \oplus (3 \ominus 2)\big)\big) \oplus \big(7 \ominus (5 \oplus 6)\big)\big)$

sweep 8: $\big((9 \ominus 8) \ominus \big(\big(4 \ominus \big(1 \oplus (3 \ominus 2)\big)\big) \oplus \big(7 \ominus (5 \oplus 6)\big)\big)\big)$

final: $\big((1 \ominus 1) \ominus \big(\big(1 \ominus \big(1 \oplus (1 \ominus 1)\big)\big) \oplus \big(1 \ominus (1 \oplus 1)\big)\big)\big) = \mathrm{sw}(\pi)$

Right Chain Condition

Proposition (Shapiro? West? Folkloric?)

For $n \geq 1$, the sweeping-algorithm $\pi \mapsto \mathrm{sw}(\pi)$ is a bijection between $\mathfrak{S}_n(2413, 3142)$ and $\mathcal{W}_n$.

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Associativity gives us $(\pi \oplus \sigma) \oplus \tau = \pi \oplus (\sigma \oplus \tau)$ and $(\pi \ominus \sigma) \ominus \tau = \pi \ominus (\sigma \ominus \tau)$. Since we always sweep from left to right in our decomposition, we will always first get block $(B_1 \oplus B_2) \oplus B_3$ before we can form $B_1 \oplus (B_2 \oplus B_3)$, similarly for operator $\ominus$.

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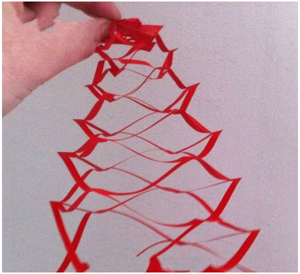

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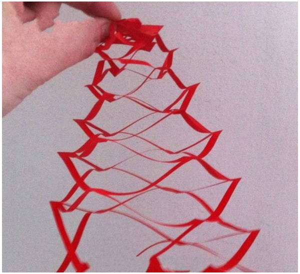

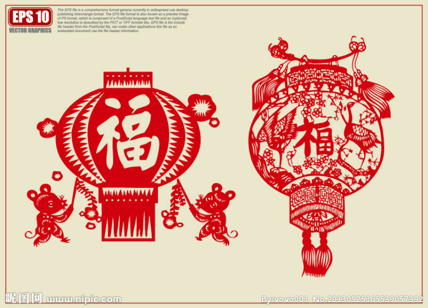

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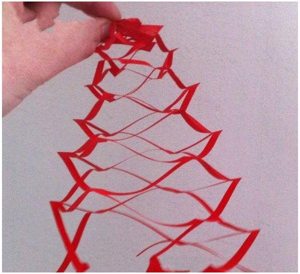

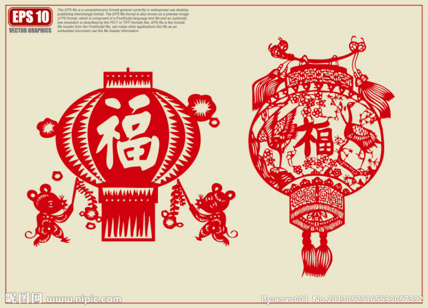

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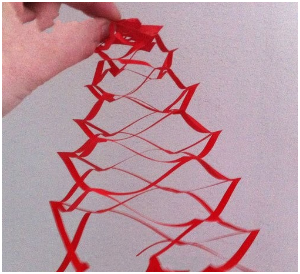

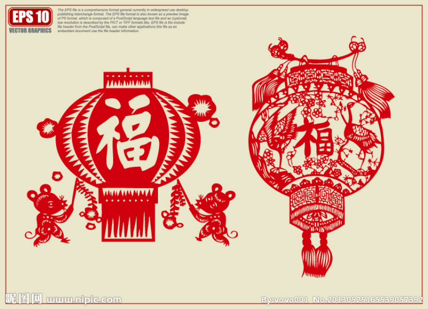

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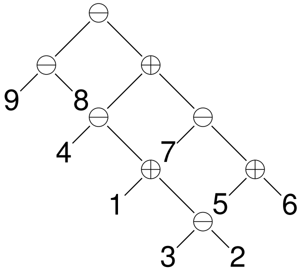
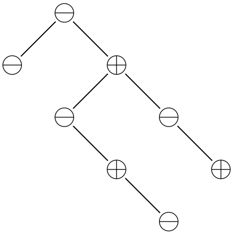
$$\mathrm{sw}(\pi) = ((1 \ominus 1) \ominus ((1 \ominus (1 \oplus (1 \ominus 1))) \oplus (1 \ominus (1 \oplus 1))))$$





Definition

Given a binary tree T, its right chain is any maximal chain composed of right children except the first node, which is either the root or a left child. The length of a right chain is the number of nodes on this chain. And the number of its right chains is denoted as $r(T)$. Similary, the number of right chains with even (resp. odd) length is denoted as $r_e(T)$ (resp. $r_o(T)$).

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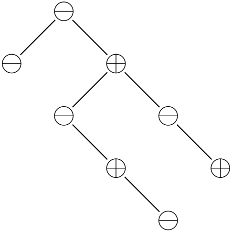
We call a binary tree di-sk tree if it is labelled with $\oplus$ and $\ominus$ and the labelling satisfies the right chain condition. The set of all di-sk trees with $n-1$ nodes is denoted as $\mathfrak{DT}_n$.

Right Chain, Lock and Hang



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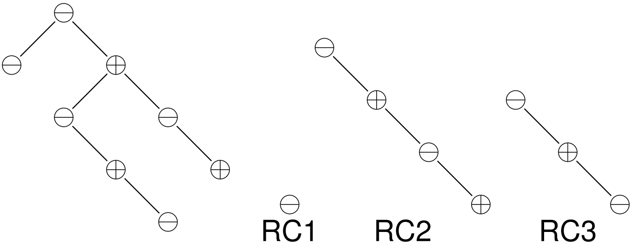

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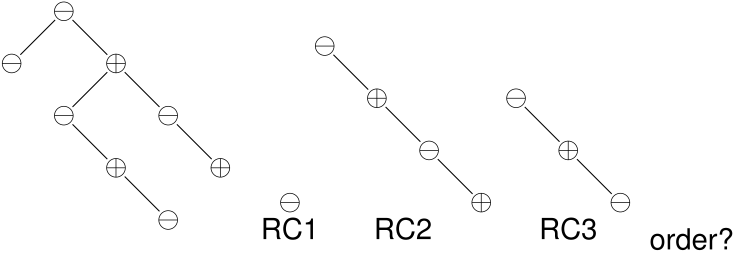

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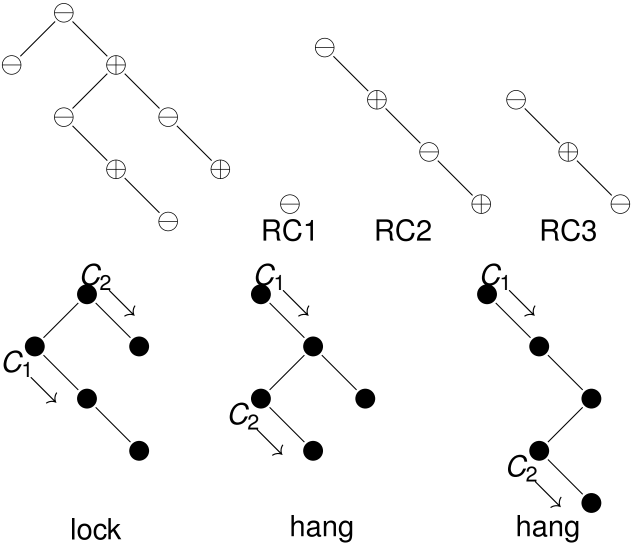

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   Back to see the example.

Outline

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$$\gamma^S_{n,k} = \#\{T \in \mathfrak{DT}_n : \text{ 1st node } \oplus, \text{ no cons. } \ominus, \# \text{ of } \ominus = k\}$$
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IV. Construct bijection between $\mathfrak{D}\mathfrak{T}^1_{n,k}$ and $\mathfrak{D}\mathfrak{T}^2_{n,k}$.

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- Idea: Cut and paste at “appropriate” places!

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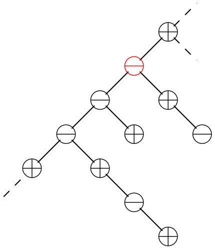

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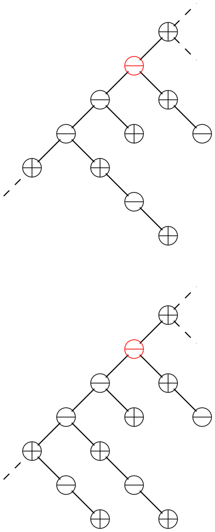

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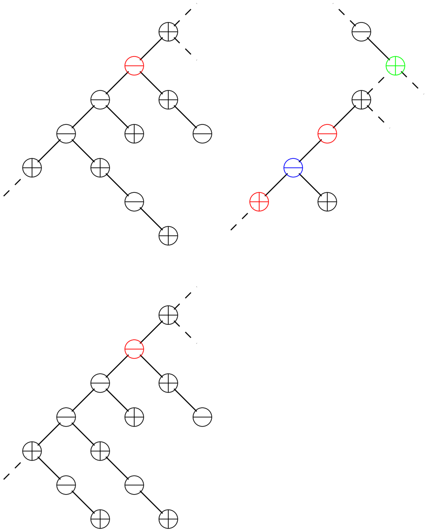

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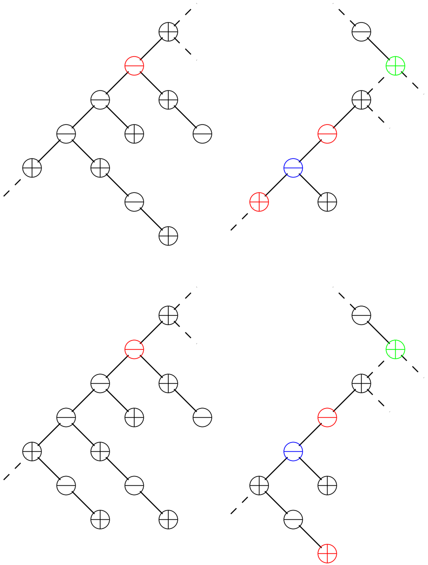

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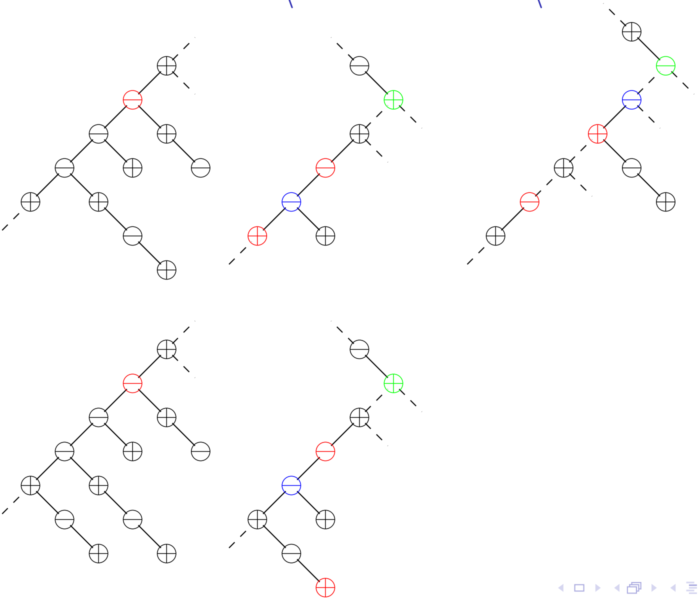

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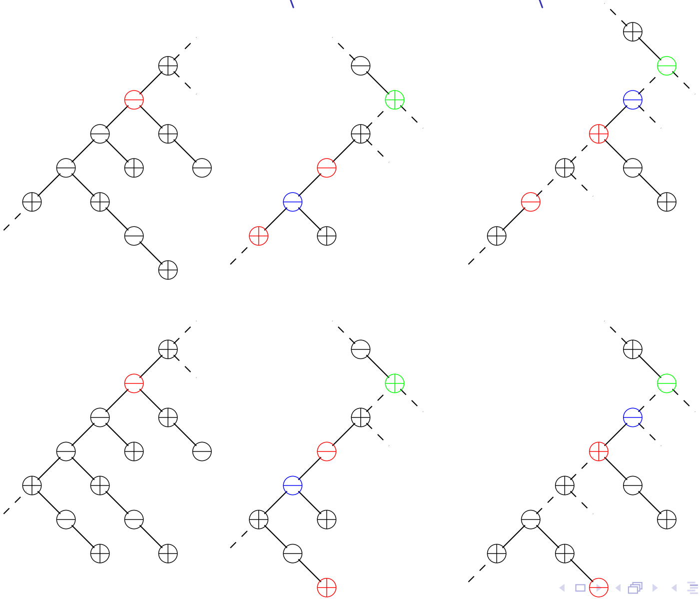

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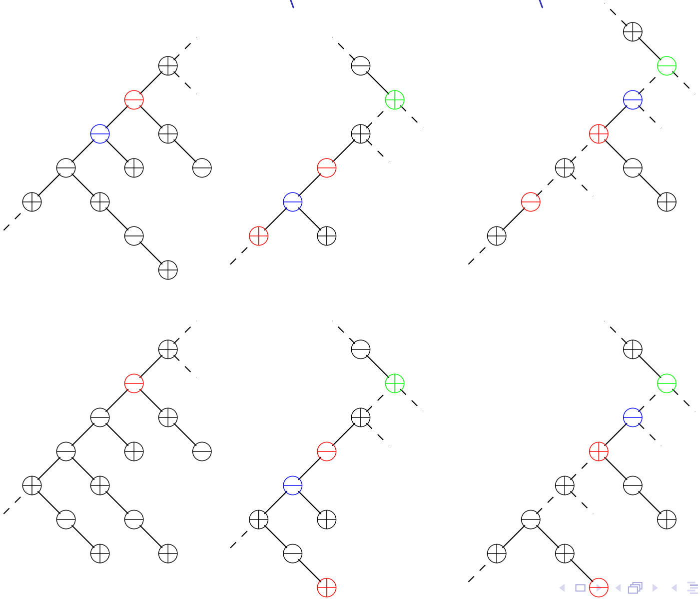

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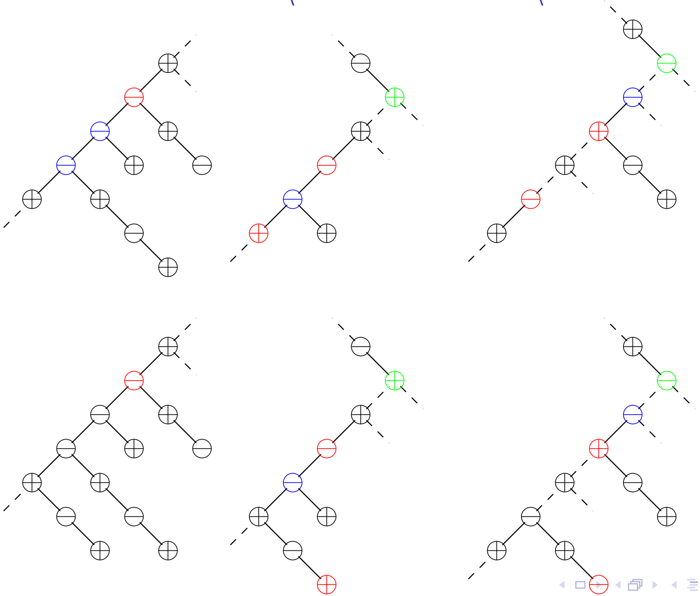

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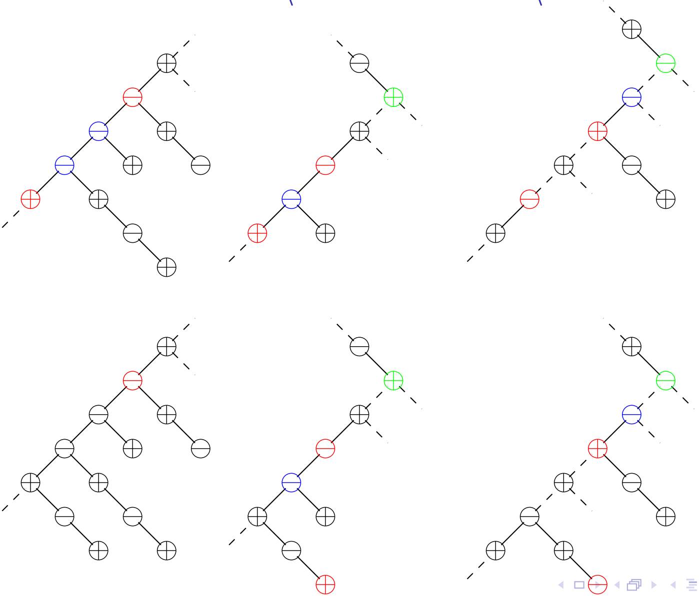

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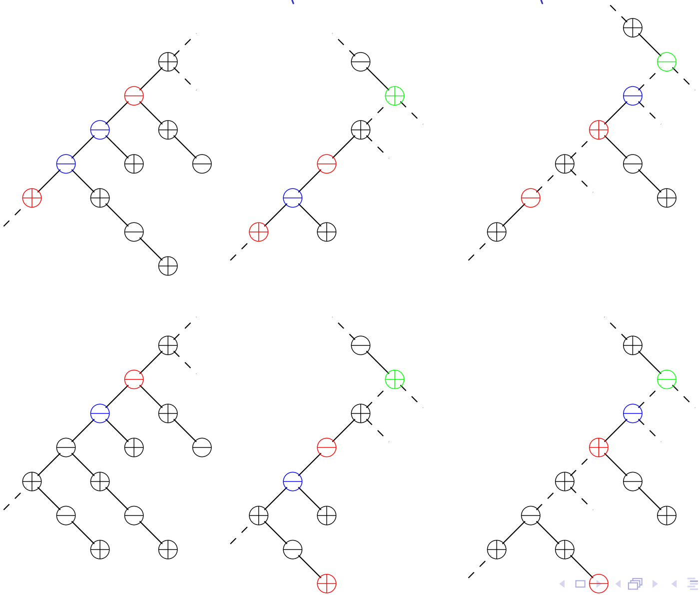
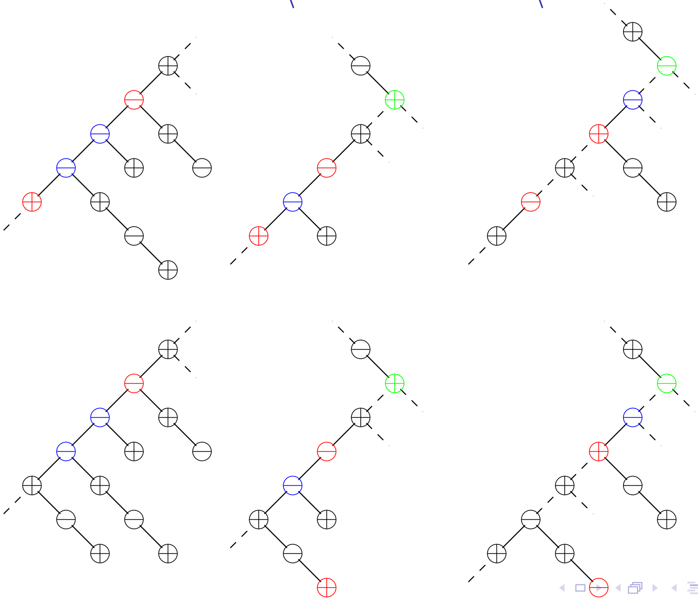
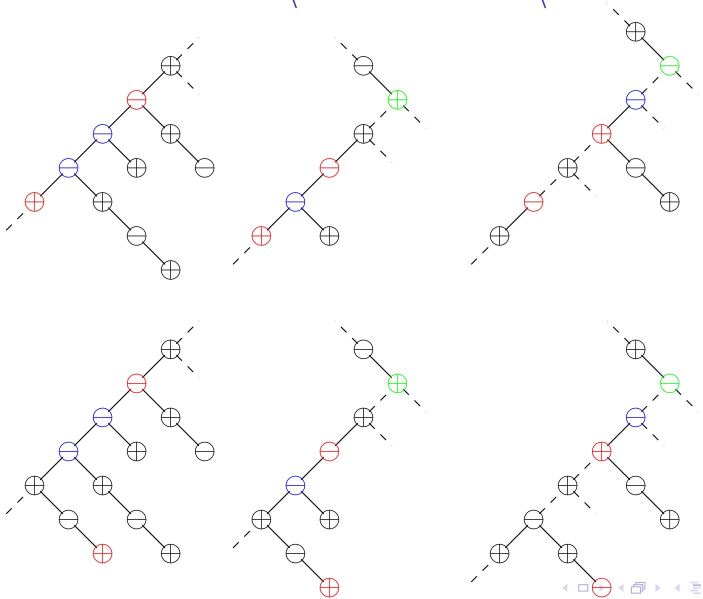


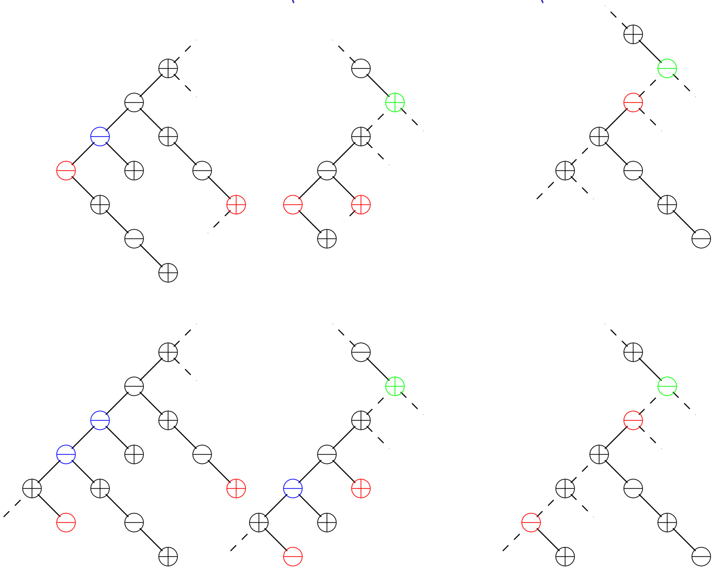
$$\mathfrak{DT}^2 \backslash \mathfrak{DT}^1 \xrightarrow{\psi} \mathfrak{DT}^1 \backslash \mathfrak{DT}^2$$



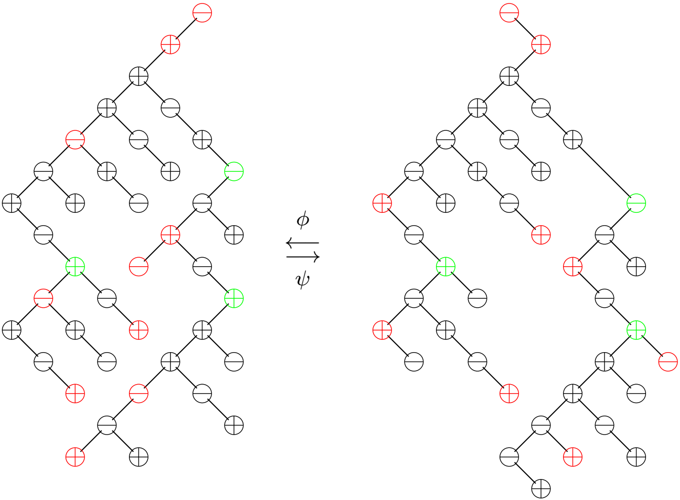
$\mathfrak{DT}^2 \backslash \mathfrak{DT}^1 \xrightarrow{\psi} \mathfrak{DT}^1 \backslash \mathfrak{DT}^2$



$$\mathfrak{DT}^1 \backslash \mathfrak{DT}^2 \xrightarrow{\phi} \mathfrak{DT}^2 \backslash \mathfrak{DT}^1$$



Example with multiple cases



Outline

In retrospect. Both $\mathfrak{S}_n(312)$ and $\mathfrak{S}_n(3142, 2413)$ “respect” $\mathfrak{S}_n$’s γ-decomposition.

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$$A_n(t) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma_{n,k}^{A} t^k (1+t)^{n-1-2k},$$

$$\gamma_{n,k}^{A} = \#\{\pi \in \mathfrak{S}_n : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\};$$

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$$N_n(t) := \sum_{\pi \in \mathfrak{S}_n(312)} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^{N} t^k (1+t)^{n-1-2k},$$

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$$\gamma^N_{n,k} = \#\{\pi\in\mathfrak{S}_n(312) : \mathrm{dd}(\pi)=0, \mathrm{des}(\pi)=k\};$$

$$S_n(t) := \sum_{\pi\in\mathfrak{S}_n(2413,3142)} t^{\mathrm{des}(\pi)} = \sum_{k\geq 0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma^S_{n,k} t^k (1+t)^{n-1-2k},$$

$$\gamma^S_{n,k} = \#\{\pi\in\mathfrak{S}_n(3142,2413) : \mathrm{dd}(\pi)=0, \mathrm{des}(\pi)=k\}.$$

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  The descent polynomials $S_n(t)$ and $D_n(t)$ are real-rooted for each $n \geq 2$.

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Thanks for listening!