Subalgebras of the descent algebra based on alternating runs

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Definition

In a permutation $\sigma(1), \sigma(2), \ldots$, we call *alternating run* a maximal monotone subsequence of consecutive values. We denote denote by $\mathrm{run}(\sigma)$ the number of alternating runs.

Example

The alternating sequences of $\sigma = 1275436$ are 127, 7543, 36
So $\mathrm{run}(\sigma) = 3$.

$1 \leq \mathrm{run}(\sigma) \leq n - 1$ for $\sigma \in \mathfrak{S}_n$, and $\mathrm{run}(\sigma) = n - 1$ for up-down or down-up alternating permutations.

Previous works on alternating sequences:
André (1881), Carlitz (1970's), Bóna & Ehrenborg (2000), Canfield & Wilf (2008), Ma (2012, 2013), Fewster & Siemssen (2014).

Let $R(n,k)$ denote the number of permutations in $\mathfrak{S}_n$ with k alternating runs, André showed

$$R(n,k) = kR(n-1,k) + 2R(n-1,k-1) + (n-k)R(n-1,k-2)$$

and Carlitz showed

$$\sum_{n\geq 0} \frac{z^n}{n!} \sum_{k=1}^{n} R(n+1,k)x^{n-k} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin(z\sqrt{1-x^2})}{x-\cos(z\sqrt{1-x^2})}\right)^2.$$

Another result (Doyle and Rockmore, Petersen): $W_k = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{run}(\sigma)=k}} \sigma$ linearly span a commutative subalgebra $\mathcal{W} \subset \mathbb{Z}[\mathfrak{S}_n]$, i.e. $W_i W_j = W_j W_i = \sum c_{i,j,k} W_k$.

Example

For $n = 3$, $W_1 = 123 + 321$, $W_2 = 132 + 213 + 231 + 312$.

$$W_1^2 = 2W_1, \qquad W_1 W_2 = W_2 W_1 = 2W_2, \qquad W_2^2 = 4W_1 + 2W_2$$

There are 2 orthogonal idempotents: $\frac{1}{6}(W_1 + W_2), \frac{1}{6}(2W_1 - W_2)$.

Context, the descent algebra

The *descent set* of a permutation $\sigma \in \mathfrak{S}_n$ is

$$\mathrm{Des}(\sigma) = \{i \ : \ \sigma(i) > \sigma(i+1)\} \subset \{1, \ldots, n-1\}.$$

For $I \subset \{1, \ldots, n-1\}$, let $R_I = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{Des}(\sigma) = I}} \sigma$. Then the R_I linearly span a subalgebra of $\mathbb{Z}[\mathfrak{S}_n]$, called *descent algebra*, denoted $\mathcal{D}_n$ [Solomon, 1976].

For example with $n = 3$, $(132 + 231)^2 = 123 + 321 + (213 + 312)$, so $R_{\{2\}}^2 = R_{\{\}} + R_{\{1,2\}} + R_{\{1\}}$.

$\mathrm{run}(\sigma)$ only depends on $\mathrm{Des}(\sigma)$, so $\mathcal{W} \subset \mathcal{D}_n$.

The Eulerian subalgebra

Let $\operatorname{des}(\sigma) = \# \operatorname{Des}(\sigma)$. Loday (1989) showed that the n elements

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}(\sigma)=k}} \sigma$$

linearly span a commutative subalgebra of $\mathcal{D}_n$.

Its n orthogonal idempotents have various applications in algebra, cf. Mielnik & Plebański (1970), Feigin & Tsygan (1987), Gerstenhaber & Schack (1987), Reutenauer (1986), Patras (1991).

It is our model and motivation to study commutative subalgebras of the descent algebra.

Peak-Eulerian algebras

Definition

A *peak* of a permutation is an integer $2 \leq i \leq n-1$ with $\sigma(i-1) < \sigma(i) > \sigma(i+1)$. We denote $\mathrm{pk}(\sigma)$ the number of peaks σ, and $\mathrm{pk}^{\circ}(\sigma) = \mathrm{pk}(\sigma) + \chi[\sigma(1) > \sigma(2)]$.

These statistics also give rise to commutative subalgebras of $\mathcal{D}_n$ having a basis of orthogonal idempotents (Schocker [2002], Petersen [2005], Aguiar-Nyman-Orellana [2006]).

In this situation, what we want to do is the following:

1. Show that W_k generate an algebra,
2. Show that it is commutative,
3. Show the existence of orthogonal idempotents ($J_1, \dots, J_{n-1}$ with $J_k J_\ell = \delta_{k\ell} J_k$),
4. Find a natural and explicit construction of the orthogonal idempotents.

(A lot of things for the run algebras can be obtained from known results on the peak algebras, but let us start from scratch.)

Warm up: Atkinson's short proof of existence of the descent algebra

Recall that $R_I = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{Des}(\sigma) = I}} \sigma$. To show that $R_I R_J = \sum_K c_{IJ}^K R_K$, we need to show that the coeff. of σ equals the coeff. of τ in $R_I R_J$ if $\mathrm{Des}(\sigma) = \mathrm{Des}(\tau)$.

Lemma

$\{\sigma : \mathrm{Des}(\sigma) = K\}$ *is an interval in* $\mathfrak{S}_n$ *with the left weak order (such that* $\sigma \lessdot s_i \sigma$ *if* $\mathrm{inv}(\sigma) + 1 = \mathrm{inv}(s_i \sigma)$*). Hence it is connected.*

Proof.

Note that $\mathrm{Des}(s_i \sigma) = \mathrm{Des}(\sigma)$ iff i and $i+1$ are not in adjacent positions. Using this we can show the existence of a unique minimal element and a unique maximal element. □

Thanks to this lemma, we need only to consider the case where $\tau = s_i \sigma$.

We can give a bijection between

- factorizations $\sigma = \alpha\beta$ where $\mathrm{Des}(\alpha) = I$, $\mathrm{Des}(\beta) = J$ (their number is the coefficient of σ in $R_I R_J$).
- factorizations $s_i\sigma = \alpha\beta$ where $\mathrm{Des}(\alpha) = I$, $\mathrm{Des}(\beta) = J$ (their number is the coefficient of $s_i\sigma$ in $R_I R_J$).

To a factorization $\alpha\beta$ we associate the factorization $(s_i\alpha)\beta$ if $\mathrm{Des}(s_i\alpha) = \mathrm{Des}(\alpha)$.

Otherwise, $s_i\alpha = \alpha s_j$. To $\alpha\beta$ we associate $\alpha(s_j\beta)$ if $\mathrm{Des}(s_j\beta) = \mathrm{Des}(\beta)$.

There is no other case to consider ! Otherwise, $s_j\beta = \beta s_k$, but it follows $s_i\sigma = \sigma s_k$, this contradicts $\mathrm{Des}(s_i\sigma) = \mathrm{Des}(\sigma)$.

The run algebra

We denote $W_k^+ = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{run}(\sigma)=k \\ \sigma(1)<\sigma(2)}} \sigma$ and $W_k^- = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{run}(\sigma)=k \\ \sigma(1)>\sigma(2)}} \sigma$.

Theorem

The elements W_k^+ and W_k^- $(1 \leq k \leq n-1)$ linearly span a subalgebra of $\mathcal{D}_n$, of $\mathcal{W}^{\pm}$.

It is not commutative, for example if $n = 3$, $W_2^+ W_2^- \neq W_2^- W_2^+$.

It contains the previous run algebra ($W_k = W_k^+ + W_k^-$) but also the peak algebras.

Atkinson's proof can be adapted to this case, but we need an extra (rather nontrivial) bijection.

To adapt Atkinson's proof, we first need:

Lemma

The permutations appearing in W_k^+ (or W_k^-) define a connected subset of $\mathfrak{S}_n$ with the left weak order.

Proof.

This subset is a union of descent class and each descent class is connected.

The descent classes are connected according to the lattice of subsets of $[1, \ldots, n-1]$ (wrt inclusion).

For example with W_4^+, the descent classes have the form $0\ldots0\,1\ldots1\,0\ldots0\,1\ldots1$. They all can be reached from $0\ldots0\,1\,0\,1$.

□

The proof

We show $W_i^+ W_j^+ \in \mathcal{W}^{\pm}$, other cases follow by symmetry.

Let σ and τ with $\mathrm{run}(\sigma) = \mathrm{run}(\tau)$, both in W_k^+ (or W_k^-). We give a bijection between

$$\{(\alpha, \beta) \ : \ \alpha\beta = \sigma, \ \alpha \in W_i^+, \ \beta \in W_j^+\}$$

and

$$\{(\alpha, \beta) \ : \ \alpha\beta = \tau, \ \alpha \in W_i^+, \ \beta \in W_j^+\}.$$

By connectedness, we can only consider the case $\tau = s_i\sigma$.

So we need a bijection Φ between

$$\{(\alpha, \beta)\ :\ \alpha\beta = \sigma,\ \alpha \in W_i^+,\ \beta \in W_j^+\}$$

and

$$\{(\alpha, \beta)\ :\ \alpha\beta = s_i\sigma,\ \alpha \in W_i^+,\ \beta \in W_j^+\}.$$

- First case: if $s_i\alpha \in W_i^+$, we define $\Phi(\alpha, \beta) = (s_i\alpha, \beta)$.

Otherwise, i and $i+1$ are neighbours in α, so $\exists g, s_i\alpha = \alpha s_g$.

- Second case: if $s_g\beta \in W_j^+$, we define $\Phi(\alpha, \beta) = (\alpha, s_g\beta)$.

Otherwise, g and $g+1$ are neighbours in β, so $\exists h, s_g\beta = \beta s_h$.

This is not finished yet...

Lemma: We have $2 \leq h \leq n-2$ and $2 \leq g \leq n-2$ (otherwise we would have been in the first or second case).

Recall that $s_i\alpha = \alpha s_g$.

Lemma: Either $\alpha(g-1) < i$ and $\alpha(g+2) > i+1$, or $\alpha(g-1) > i+1$ and $\alpha(g+2) < i$

(because $\alpha \in \mathcal{W}_j^+$ but $s_i\alpha \notin \mathcal{W}_j^+$, then look at the factor $u\, i\, i+1\, v$ or $u\, i+1\, i\, v$ in α).

$+$ two similar lemmas, one for β, one for σ. There are 8 cases (or maybe 16 if we are not careful).

Then the bijection consists in decomposing α in 4 blocks and reassembling the blocks while exchanging i and $i+1$ in the process. Details omitted.

Knowing that $\mathcal{W}^{\pm}$ is an algebra, we deduce that $\mathcal{W}$ is too, more precisely $\mathcal{W}$ is the ideal $W_1\mathcal{W}^{\pm}$ because $W_1 W_j^{+} = W_1 W_j^{-} = W_j$.

Commutativity of $\mathcal{W}$

$W_i W_j = W_j W_i$ could be proved by a bijection between

$$\{(\alpha, \beta) \ : \ \alpha\beta = \sigma, \ \mathsf{run}(\alpha) = i, \ \mathsf{run}(\beta) = j\}$$

and

$$\{(\alpha, \beta) \ : \ \alpha\beta = \sigma, \ \mathsf{run}(\alpha) = j, \ \mathsf{run}(\beta) = i\}.$$

But it is a difficult problem and have only an algebraic proof.

Let Sym be the algebra of symmetric function and Sym_n its homogeneous component of degree n.
Sym_n is a commutative algebra for the internal product $*$ defined by

$$p_\lambda * p_\lambda = z_\lambda p_\lambda, \qquad p_\lambda * p_\mu = 0 \text{ if } \lambda \neq \mu$$

where $z_\lambda = \prod_i i^{m_i} m_i!$, m_i being the multiplicity of i in λ.

Proposition (Solomon 1976)

There is a morphism $\Gamma : \mathcal{D}_n \to Sym_n$ defined by

$$\Gamma(R_I) = s_{\lambda/\mu},$$

the skew Schur function where λ/μ is the ribbon defined by: if $I = \{i_1, i_2, i_3 \dots\}$, then $\lambda = (\dots, i_3 - 2, i_2 - 1, i_1)$.

Proposition

Γ realizes an isomorphism from $\mathcal{W}$ to $\Gamma(\mathcal{W})$. In particular, $\mathcal{W}$ is commutative.

It is a consequence of:

Proposition

The kernel of $\Gamma : \mathcal{W}^{\pm} \to \Gamma(\mathcal{W}^{\pm})$ is generated by the elements $W_{2k}^{+} - W_{2k}^{-}$ ($1 \leq k < n/2$).

Proof.

It contains these elements since $\Gamma(W_{2k}^{-}) = \Gamma(\omega W_{2k}^{+} \omega) = \Gamma(W_{2k}^{+})$, where $\omega = W_1^{-} = n \dots 321$.

So its dimension is at least $\lceil \frac{n}{2} \rceil - 1$. It remains to show that the dimension of the image is $\lfloor \frac{3n}{2} \rfloor - 1$, the sum being $2n - 2$. □

Proposition

A subalgebra of Sym_n has a basis of orthogonal idempotents $\sum_{\lambda \in X_1} \frac{1}{z_\lambda} p_\lambda$, $\sum_{\lambda \in X_2} \frac{1}{z_\lambda} p_\lambda$, ... where $X_1, X_2, \ldots$ are disjoint subsets of the set of partitions of n.

Proposition

The orthogonal idempotents of $\Gamma(\mathcal{W}^{\pm})$, are

$$\sum_{\substack{\lambda \vdash n \\ \ell_o(\lambda)=k, \\ \ell_e(\lambda)=0}} \frac{1}{z_\lambda} p_\lambda, \qquad \sum_{\substack{\lambda \vdash n \\ \ell_o(\lambda)=k, \\ \ell_e(\lambda) \text{ odd}}} \frac{1}{z_\lambda} p_\lambda, \qquad \sum_{\substack{\lambda \vdash n \\ \ell_o(\lambda)=k, \\ \ell_e(\lambda)>0 \text{ even}}} \frac{1}{z_\lambda} p_\lambda.$$

where $\ell_o(\lambda)$ (resp. $\ell_e(\lambda)$) is the number of odd (resp. even) parts of λ.

So $\dim(\Gamma(\mathcal{W}^{\pm})) = \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor + (\lfloor \frac{n}{2} \rfloor - 1) = \lfloor \frac{3n}{2} \rfloor - 1.$

This is obtained by computing $\langle \Gamma(x) | p_\lambda \rangle$ for well chosen elements $x \in \mathcal{W}$ and checking that the results only depends on

- $\ell_o(\lambda)$
- $\ell_e(\lambda) \mod 2$,
- whether $\ell_e(\lambda)$ is positive or not.

Remark: In the case of the Eulerian algebra $\mathcal{E}$ generated by $\sum_{\mathrm{des}(\sigma)=k} \sigma$, the orthogonal idempotents of $\Gamma(\mathcal{E})$ are $\sum_{\ell(\lambda)=k} \frac{1}{z_\lambda} p_\lambda$.

This is a reformulation of the fact that the *Foulkes character* $\chi_{n,k}$ (that will not be defined here) evaluated on a permutation $\sigma \in \mathfrak{S}_n$, only depends on its number of cycle.

Hopf algebras

Further results rely on the Hopf algebra of permutations known as the Malvenuto-Reutenauer algebra or *free quasi-symmetric functions*. It is defined on a basis $\mathbf{F}_\sigma$. and the product is the *shifted shuffle*: if $\alpha \in \mathfrak{S}_k$, $\beta \in \mathfrak{S}_\ell$,

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum \mathbf{F}_\sigma$$

summed over shuffles σ of $\alpha_1 \dots \alpha_k$ and $\beta_1 + k \dots \beta_\ell + k$.

Example: $\mathbf{F}_{21}\mathbf{F}_{231} = F_{21453} + F_{24153} + F_{42153} + F_{24513} + \dots$

Internal product: $\mathbf{F}_\alpha \star \mathbf{F}_\beta = \mathbf{F}_{\alpha\beta}$ if they have the same degree, 0 otherwise. (So in degree n this gives the symmetric group algebra $\mathbb{Z}[\mathfrak{S}_n]$).

The coproduct is *standardized deconcatenation*:

$$\Delta(\mathbf{F}_\sigma) = \sum_{i=0}^{n} \mathbf{F}_{\mathrm{std}(\sigma(1)\ldots\sigma(i))} \otimes \mathbf{F}_{\mathrm{std}(\sigma(i+1)\ldots\sigma(n))}$$

where $\mathrm{std}(a_1 \ldots a_i)$ is the unique permutation with the same inversion set as $a_1 \ldots a_n$.

The elements $\mathbf{R}_I$ (that perhaps should be denoted $\mathbf{R}_{(n,I)}$) where I is a subset of $\{1, \dots, n-1\}$ are defined by

$$\mathbf{R}_I = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{Des}(\sigma^{-1}) = I}} \mathbf{F}_\sigma,$$

and they span a Hopf subalgebra **NSym** known as *noncommutative symmetric functions* (Gelfand et al.).
And $(\mathbf{NSym}_n, \star)$ is anti isomorphic to the descent algebra.

In this subalgebra we have the *splitting formula* :

$$(xy) \star z = \sum (x \star z_1)(y \star z_2)$$

where we sum over all terms $z_1 \otimes z_2$ appearing in $\Delta(z)$ (Sweedler's notation).

How to use the Hopf algebra structure to get a subalgebra of the descent algebra ? Let us review the case of the Eulerian subalgebra. Let

$$\mathbf{E} = \sum_{n \geq 0} \mathbf{F}_{12\ldots n}.$$

Proposition $\mathbf{E}^k \star \mathbf{E}^\ell = \mathbf{E}^{k\ell}$.

Proof. First $\Delta(\mathbf{E}) = \mathbf{E} \otimes \mathbf{E}$, consequently $\Delta(\mathbf{E}^\ell) = \mathbf{E}^\ell \otimes \mathbf{E}^\ell$, then use the splitting formula.

Corollary Let π_n denote the projection on the degree n homogenous component, then $\pi_n(\mathbf{E}^k)$ for $k \geq 1$ linearly span a subalgebra of the descent algebra.
It turns out to be the Eulerian subalgebra $\mathcal{E}$.

To deal with the case of the run algebra, let also

$$\tilde{\mathbf{E}} = \sum_{n \geq 0} \mathbf{F}_{n\ldots 21}$$

Then, consider

$\mathbf{V}_k^+ = \mathbf{E}\tilde{\mathbf{E}}\mathbf{E}\ldots$ (k factors, the last one depends on the parity of k),
$\mathbf{V}_k^- = \tilde{\mathbf{E}}\mathbf{E}\tilde{\mathbf{E}}\ldots$ (idem).

Proposition

$$\begin{array}{lll} \mathbf{V}_k^+ \star V_\ell^+ = \mathbf{V}_{k\ell}^+, & \mathbf{V}_k^+ \star \mathbf{V}_\ell^- = \mathbf{V}_{k\ell}^- & \\ \mathbf{V}_k^- \star V_\ell^+ = \mathbf{V}_{k\ell}^+, & \mathbf{V}_k^- \star \mathbf{V}_\ell^- = \mathbf{V}_{k\ell}^- & \textit{if } \ell \textit{ is even,} \\ \mathbf{V}_k^- \star V_\ell^+ = \mathbf{V}_{k\ell}^-, & \mathbf{V}_k^- \star \mathbf{V}_\ell^- = \mathbf{V}_{k\ell}^+ & \textit{if } \ell \textit{ is odd.} \end{array}$$

Then, $\pi_n(\mathbf{V}_k^+)$ and $\pi_n(\mathbf{V}_k^-)$ for $1 \leq k \leq n-1$ give another basis of $\mathcal{W}^\pm$.

Thanks for your attention