On the Turan-type Problems for Hypergraph

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Simple example

- Extremal Combinatorics: It deals with the problems of determining or estimating the maximum or minimum possible cardinality of a collection of finite objects (Graphs, Hypergraphs, Sets, Numbers, ...) that satisfies certain requirements.

Example

What is the maximum number of edges in a triangle-free graph on n vertices ?

- Let $x, y \in V(G)$ with $(x, y) \in E(G)$.
- For $z \in V(G) - \{x, y\}$, z is adjacent to at most one of x, y.
  $\implies (d(x) - 1) + (d(y) - 1) \leqslant n - 2$. $(d(x) + d(y) \leqslant n)$
- Sum this over all edges (x, y).

$$\frac{4}{n}|E(G)|^2 = \frac{1}{n}(\sum_x d(x))^2 \leqslant \sum_x d^2(x) = \sum_{x,y \in E(G)} (d(x) + d(y)) \leqslant n|E(G)|$$

$$\implies |E(G)| \leqslant \frac{n^2}{4}$$

What is Turan number of Hypergraph?

- Hypergraph (a generalization of a graph) : a pair (V, E) where V:= set of vertices and E:= set of non empty subset of V (hyperedge)

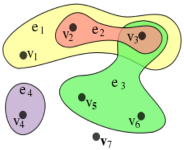

Figure: Hypergraph

Turan type question

What is the maximum number of hyperedges in a hypregraph if the hypergraph contains no F as a subhypergraph (F-free)?
This maximum is denoted as $ex(n, F)$ (extremal number or turan number of F).

- an edge of n-vertex hypergraph $H \iff$ a subset of $[n]$ in a set family $\mathcal{F}$.
- Turan type problems of the hypergraph H are same as studying the maximum size of a set family $\mathcal{F}$ of distinct subsets of an n-element set that satisfies certain given conditions.

Certain properties on Sets

- Union-free, B_2-free, Intersecting, t-intersecting, cross t-intersecting
- A family F of sets is called *intersecting* for every $F, F' \in \mathcal{F}$ such that

$$F \cap F' \neq \phi.$$

Q: How large can such an family be? $\Longrightarrow \mathcal{F} \leqslant 2^{n-1}$.
(Take all the subsets containing a fixed element.
This is an intersecting family with $|\mathcal{F}| = 2^{n-1}$.)

- A family of sets is called *union-free* if there are no three distinct sets G_1, G_2, G_3 such that

$$G_1 \cup G_2 = G_3.$$

- A family of sets is called B_2*-free* if there are no four distinct sets G_1, G_2, G_3, G_4 such that

$$G_1 \cup G_2 = G_3 \;\; and \;\; G_1 \cap G_2 = G_4.$$

Erdős and Shelah Conjectures

- A family of sets is called *union-free* if there are no three distinct sets G_1, G_2, G_3 such that $G_1 \cup G_2 = G_3$.

Problem 1 (Erdős and Komlós, 1969)

How large of a union-free subfamily does every family of m sets have?
Denote this number by $f(m, \text{union-free})$.

Theorem (Erdős and Shelah, 1972)

$\sqrt{2m} - 1 < f(m, \text{union-free}) < 2\sqrt{m} + 1.$

Conjecture 1 (Erdős and Shelah, 1972)

$$f(m, \text{union-free}) = (2 + o(1))\sqrt{m}.$$

$\Longrightarrow$ Solved by Fox, Lee, Sudakov (2012).

Erdős and Shelah Conjectures

- A family of sets is called B_2*-free* if there are no four distinct sets G_1, G_2, G_3, G_4 such that $G_1 \cup G_2 = G_3$ and $G_1 \cap G_2 = G_4$.

Question (Erdős and Shelah, 1972)

How large of a B_2-free subfamily does every family of m sets have?
Denote this number by $f(m, B_2\text{-free})$.

Theorem (Erdős and Shelah, 1972)

$$f(m, B_2\text{-free}) \leqslant \frac{3}{2} m^{2/3}.$$

Conjecture 2 (Erdős and Shelah, 1972)

$$f(m, B_2\text{-free}) > c_2 m^{2/3}.$$

$\Longrightarrow$ Verified by Barát, Füredi, Kantor, Kim, Patkos (2012)
—— By using the Probabilistic Method (Deletion Method)

Simple Example of Deletion Method

- A set of vertices is independent if no two vertices from this set are joined by edge.
- independence number $\alpha(G)$:= maximum number of vertices in G with no edges between them.

Example (Spencer 1987)

If a graph $G = (V, E)$ has n vertices and $nk/2$ edges then $\alpha(G) \geqslant n/2k$.

1. Consider a random subset $S \subset V$ of vertices
   (Select every vertex in S independently probability p).
2. X := random variable denoting number of vertices in S.
   Y := random variable denoting number of edges, both ends of which lie in S.
3. Delete one vertex from each edge from S.
4. We obtain a independent set S' of size at least $X - Y$.
5. $$np - \frac{nk}{2}p^2 = E(X) - E(Y) = E(X - Y) \leqslant \alpha(G).$$
6. Choose $p = 1/k$ to maximize this quantity, giving $E(X - Y) = \frac{n}{k} - \frac{n}{2k} = \frac{n}{2k}$.

Result

Theorem– Barát, Füredi, Kantor, Kim, Patkos (2012)

Erdős and Shelah Conjecture(1972) is true.

$$f(m, B_2\text{-free}) > c_2 m^{2/3}.$$

1. Consider a random subfamily $\mathcal{F}'$.
   (Select every set in $\mathcal{F}'$ independently with probability p).
2. $X :=$ random variable denoting number of sets in $\mathcal{F}'$.
   $Y :=$ random variable denoting the number of copies of B_2 in $\mathcal{F}'$
3. Remove a set from each subfamily in $\mathcal{F}'$ forming B_2.
4. We obtain a B_2-free subfamily $\mathcal{F}''$ of size at least $X - Y$.
5. Since $\mathbb{E}(X) = pm$, $\mathbb{E}(Y) \leqslant \binom{m}{2} p^4$,

$$mp - p^4 \binom{m}{2} \leqslant \mathbb{E}(X - Y) \leqslant f(m, B_2\text{-free}).$$

6. Put $p = 2^{-1/3} m^{-1/3}$ yields the lower bound.

$\Longrightarrow$ This result verifies a Conjecture of Erdős and Shelah from 1972.

Erdős-Ko-Rado Theorem

- A family F of sets is called *intersecting* for every $F, F' \in \mathcal{F}$ such that $F \cap F' \neq \phi$.
- Let $\mathcal{F}$ be an intersecting family of k-element subsets of $[n]$.
  or (k-uniform intersecting family of $[n]$)

Erdős-Ko-Rado Theorem (1961)

If $n \geqslant 2k$ and $\mathcal{F}$ is a k-uniform intersecting family of $[n]$ then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1}.$$

- A family $\mathcal{F}$ of sets is called *t-intersecting* if for every $F, F' \in \mathcal{F}$, $|F \cap F'| \geqslant t$.

Generalized Erdős-Ko-Rado Theorem (Frankl (1978), Wilson (1984))

If $n \geqslant (t+1)(k-t+1)$ and $\mathcal{F}$ is a k-uniform t-intersecting family of $[n]$ then

$$|\mathcal{F}| \leqslant \binom{n-t}{k-t}.$$

History

- A family $\mathcal{F}$ of sets is called *t-intersecting* if for every $F, F' \in \mathcal{F}$, $|F \cap F'| \geqslant t$.
- A family $\mathcal{F}$ of sets is called *L-intersecting* if for every $F, F' \in \mathcal{F}$,

$$|F \cap F'| \in L = \{l_1, l_2, \cdots, l_s\}.$$

Ray-Chaudhuri and Wilson (1975)

If $\mathcal{F}$ is a k-uniform L-intersection family of subsets of $[n]$, then $|\mathcal{F}| \leqslant \binom{n}{s}$, where $L = \{l_1, l_2, \cdots, l_s\}$.

Deza, Erdős and Frankl (1978)

If $\mathcal{F}$ is a k-uniform L-intersecting family of subsets of $[n]$, then $|\mathcal{F}| \leqslant \prod_{i=1}^{s} \frac{n-l_i}{k-l_i}$, where $L = \{l_1, l_2, \cdots, l_s\}$.

Frankl-Wilson (1981) [Modular version of Ray-Chaudhuri-Wilson]

If $\mathcal{F}$ is a k-uniform family of subsets of $[n]$ s.t. $k = l_0 \ (mod\ p) \ \notin L$ and $|F_i \cap F_j| \ (mod\ p) \ \in L = \{l_1, l_2, \cdots, l_s\}$ for every $F_i, F_j \in \mathcal{F}$ then $|\mathcal{F}| \leqslant \binom{n}{s}$, where $l_0, l_1, \cdots, l_s$ are distinct residues mod p.

Alon-Babai-Suzuki's Conjecture

Noga Alon, Babai, and Suzuki (1991)

$K = \{k_1, k_2, \cdots, k_r\}$ and $L = \{l_1, l_2, \cdots, l_s\}$ be two disjoint subsets of $\{0, 1, \cdots, p-1\}$ where p is a prime. $\mathcal{F}$ is a family of subsets of X s.t. $|F_i| \pmod p \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \pmod p \in L$ for $i \neq j$.
If $r(s-r+1) \leqslant p-1$ and $n \geqslant s + \max_{1 \leqslant i \leqslant r} k_i$, then

$$|\mathcal{F}| \leqslant \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

Alon-Babai-Suzuki's Conjecture (1991)

$K = \{k_1, k_2, \cdots, k_r\}$ and $L = \{l_1, l_2, \cdots, l_s\}$ be two disjoint subsets of $\{0, 1, \cdots, p-1\}$ where p is a prime. $\mathcal{F}$ is a family of subsets of X s.t. $|F_i| \pmod p \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \pmod p \in L$ for $i \neq j$.
If $n \geqslant s + \max_{1 \leqslant i \leqslant r} k_i$, then

$$|\mathcal{F}| \leqslant \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

Result

Theorem– Hwang and Kim (2015)

Alon-Babai-Suzuki's Conjecture (1991) is True.

- Snevily (1994) — This is True when n is sufficiently large.
- Qian and Ray Chaudhuri (2000) — This is True if $n \geqslant 2s - r$.
- Hwang et al. (2010) : some result in the non modular version. They proved if $n \geqslant s + \max_{1 \leqslant i \leqslant r} k_i$, then $|\mathcal{F}| \leqslant \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.
- Hwang and Kim (2015) verified this Conjecture
- by using Linearly independent polynomial Method.

Linearly independent polynomial Method

Linearly independent polynomial Method

- One of the most powerful methods for counting the number of sets in a certain combinatorial structure.
- In this method, we correspond polynomials to the sets.
- $\Longrightarrow$ prove that these polynomials are linearly independent in some space.
- This method has been used to prove many intersection theorems in set family
- — by Blokhuis, Alon, Babai, Suzuki, Ramanan, Snevily, Furedi, and others.

Simple Example of Linearly independent polynomial Method

Deza, Frankl, and Singhi (1983)

If $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ is a family of subsets of $[n]$ s.t. for all i, $|F_i| \pmod p \notin L$ and $|F_i \cap F_j| \pmod p \in L = \{l_1, l_2, \cdots, l_s\}$ for every $F_i, F_j \in \mathcal{F}$ then $|\mathcal{F}| \leqslant \sum_{i=0}^{s} \binom{n}{i}$.

1. Define the characteristic vector $v_i = (v_{i_1}, v_{i_2}, \cdots, v_{i_n})$ of F_i
   s.t. $v_{i_j} = 1$ if $j \in F_i$ & $v_{i_j} = 0$ if $j \notin F_i$.
   x:= n-tuple of variables $x_1, x_2, \cdots, x_n$,where $x_i = 0$ or 1.
2. Consider Polynomial $f_i(x) = \prod_{k=1}^{s}(v_i \cdot x - l_k)$. (Multilinear)
3. $\Longrightarrow f_i(v_i) \neq 0 \pmod p$ & $f_i(v_j) = 0 \pmod p$ for $i \neq j$.
   $\Longrightarrow \{f_i(x) | 1 \leqslant i \leqslant m\}$ is a linearly independent family.
4. Since all polynomials $\{f_i(x) | 1 \leqslant i \leqslant m\}$ have a degree $(\leqslant s)$,
   $\Longrightarrow m \leqslant \sum_{i=0}^{s} \binom{n}{i}$.

Proof of Alon-Babai-Suzuki's Conjecture

- $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$.
- Define the characteristic vector $v_i = (v_{i_1}, v_{i_2}, \cdots, v_{i_n})$ of F_i s.t. $v_{i_j} = 1$ if $j \in F_i$ & $v_{i_j} = 0$ if $j \notin F_i$.
- x:= n-tuple of variables $x_1, x_2, \cdots, x_n$,where $x_i = 0$ or 1.
- Consider Polynomial $f_i(x) = \prod_{j=1}^{s}(v_i \cdot x - l_j)$.
- $\Longrightarrow f_i(v_i) \neq 0 \pmod{p}$ for $1 \leqslant i \leqslant m$ &
  $f_i(v_j) = 0 \pmod{p}$ for $1 \leqslant j \neq i \leqslant m$.
- Elements of $K = \{k_1, k_2 \cdots, k_r\}$ are arranged in increasing order, that is, $k_1 = \min_{1 \leqslant i \leqslant r} k_i$ & $k_r = \max_{1 \leqslant i \leqslant r} k_i$.
- Divide n into the following four cases according to the relation of $p + k_1$ and $s + k_r$.

1. $s + k_r \leqslant n < p + k_1$
2. $s + k_r < p + k_1 \leqslant n$
3. $(s - r + 1) + k_r < p + k_1 \leqslant s + k_r \leqslant n$
4. $p + k_1 \leqslant (s - r + 1) + k_r \leqslant s + k_r \leqslant n$

Case1, Case2, Case 3:

- $\mathcal{E} = \{E_1, E_2, \cdots, E_e\}$: family of subsets of $[n]$ with size at most $s-r$, where $e = \sum_{i=0}^{s-r} \binom{n}{i}$.
- $h(x) = \prod_{k_i \in K}(x_1 + x_2 + \cdots + x_n - k_i)$.
- **Claim:** Polynomials $\{f_i(x), h(x)\prod_{j \in E_i} x_j\}$ are linearly independent over the finite field $\mathbb{F}_p$.
- $\Longrightarrow |\mathcal{F}| + \sum_{i=0}^{s-r} \binom{n}{i} \leqslant \sum_{i=0}^{s} \binom{n}{i}$.
- $\Longrightarrow |\mathcal{F}| \leqslant \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

Case 4: Let $1 \leqslant \delta \leqslant s-3r$ and $1 \leqslant \alpha \leqslant r$.

- **Claim I:** $\binom{n}{s-2r-\delta-a} + \binom{n}{2s-r-\delta-a} \leqslant \binom{n}{s}$, $0 \leqslant a < s-2r-\delta$.
  1. $\mathbf{0 \leqslant a \leqslant \frac{s-r-\delta}{2} - 1}$
  2. $\mathbf{\frac{s-r-\delta}{2} - 1 < a < s - 2r - \delta}$
- **Claim II:** $|\mathcal{F}| \leqslant (r-\alpha)\binom{n}{s} + \sum_{i=1}^{\alpha}\left(\binom{n}{s-2r-\delta-a_i} + \binom{n}{2s-r-\delta-a_i}\right) \leqslant r\binom{n}{s}$.
- **Claim III:** $\min\left[\binom{n}{s}, \binom{n}{s-1}, \cdots, \binom{n}{s-r+1}\right] = \binom{n}{s}$.
- **Claim IV:** $|\mathcal{F}| \leqslant \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

Other Results related to turan type problems for hypergraph

Non-uniform version of the EKR theorem

If $n \geqslant 2k$ and $\mathcal{F} \subset \binom{[n]}{\leqslant k}$ is a intersecting family, $|\mathcal{F}| \leqslant \binom{n-1}{k-1} + \binom{n-1}{k-2} + \ldots + \binom{n-1}{0}$.

A strengthening of EKR- Alon, Aydinian, Huang (2014)

If $n \geqslant 2k$ and $\mathcal{F} \subset \binom{[n]}{\leqslant k}$ s.t. for any $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$, $|A| + |B| \leqslant k$ holds, $|\mathcal{F}| \leqslant \binom{n-1}{k-1} + \binom{n-1}{k-2} + \ldots + \binom{n-1}{0}$.

Non-uniform version of Generalized EKR theorem

If $n \geqslant (t+1)(k-t+1)$ and $\mathcal{F} \subset \binom{[n]}{\leqslant k}$ is a t-intersecting family, $|\mathcal{F}| \leqslant \binom{n-t}{k-t} + \binom{n-t}{k-1-t} + \ldots + \binom{n-t}{0}$.

A strengthening of generalized EKR-(Kang, Kim, and Kim 2015)

If $n \geqslant (t+1)(k-t+1)$ and $\mathcal{F} \subset \binom{[n]}{\leqslant k}$ s.t. for any $A, B \in \mathcal{F}$ with $|A \cap B| < t$, $|A \triangle B| \leqslant k - t$ holds, $|\mathcal{F}| \leqslant \binom{n-t}{k-t} + \binom{n-t}{k-1-t} + \ldots + \binom{n-t}{0}$.

1. When $t = 1$, this is done by Alon, Aydinian, Huang (2014).
2. We extend Alon, Aydinian, Huang's result to all $t \geqslant 1$. (Kang, Kim, Kim (2015))

Other Results related to turan type problems for hypergraph

A graph G is said to be *H-saturated* if
– it does not contain H as a subgraph, but
– the addition of any new edge (from $E(\overline{G})$) creates a copy of H.

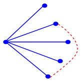

Figure: A K_3-saturated graph

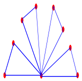

Figure: A C_4-saturated graph

The minimum number of edges in a H-saturated graphs on n vertices is denoted by the saturation number for H.

Question of this Talk:

What is the saturation number for the cycle, C_k ?

- has been considered by various authors ($k = 3$ (1964), $k = 4$ (1972,1989), $k = 5$ (2009), All k (1996, 2006)) $\Longrightarrow$ We give relatively tight bounds.

Theorem (Füredi and Kim (2013))

For $n \geqslant k \geqslant 5$,

$$sat(n, C_k) = n + \frac{n}{k} + O(\frac{n}{k^2} + k^2)$$

Ongoing and Future Work

- Problem 1:
  Find a new short proof of generalized EKR Theorem by using following property : The l-shadow of a set family $\mathcal{F}$ $(\sigma_l(\mathcal{F}))$ is defined as the family of l-subsets of its member. In 1964, Katona proved that if $\mathcal{F}$ is a family of a-element sets such that $|F \cap F'| \geqslant b \geqslant 0$ for all $F, F' \in \mathcal{F}$ then $|\mathcal{F}| \leqslant |\sigma_{a-b}\mathcal{F}|$.
- Problem 2 :
  For $r \geqslant 2$, there exists k_r, n_k s.t. for all $k > k_r$ and $n > n_k$ and $r|k$, if H is a k-uniform hypergraph with no r-regular subgraph, then $|H| \leqslant \binom{n-1}{k-1}$ and equality holds if and only if H is a full-k-star.
  $(ex(n, F) \leqslant \binom{n-1}{k-1}$ where F is r-regular subgraph?)
- Problem 3 :
  For every positive integer r, there exists k_r, n_k and $g(r)$ with the following holds. For $k \geqslant k_r$, $n \geqslant n_k$, an n-vertex k-uniform hypergraph H contains more than $g(r)\binom{n-1}{k-1}$ edges, then it contains distinct edges $A_1, B_1, \cdots, A_r, B_r$
  so that $A_i \cap B_i = \emptyset$ for all $i = 1, 2, \cdots, r$ and $A_1 \cup B_1 = A_2 \cup B_2 = \cdots = A_r \cup B_r$.
  (Property F) $(ex(n, F) \leqslant g(r)\binom{n-1}{k-1}$ where F is above property?)

Thank You!