Logarithmic Tree Numbers for Acyclic Complexes

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Seoul National University

2015 Combinatorics Workshop
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NIMS, CAMP, Daejeon

Joint work with K. Lee

Topics

- Temperley's tree number formula (Temperley 1964)

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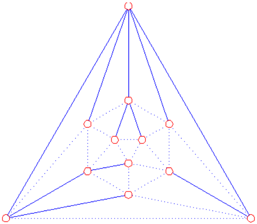
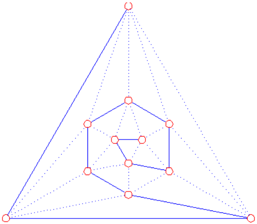
Spanning trees of a graph





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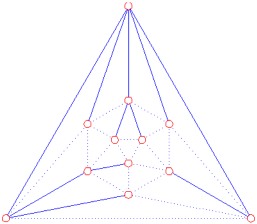

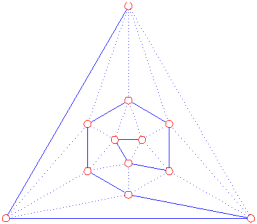

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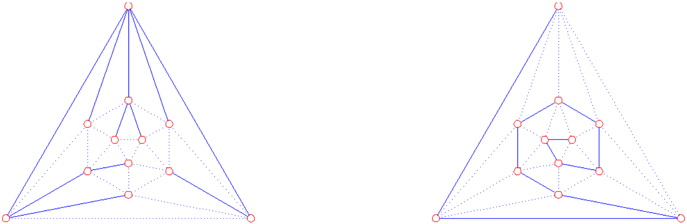

- A spanning tree B of a connected finite graph G with n vertices:
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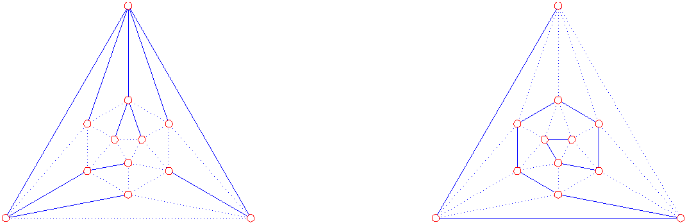

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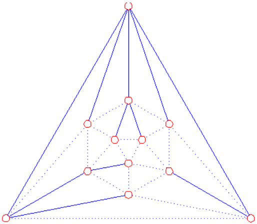

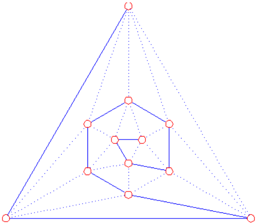

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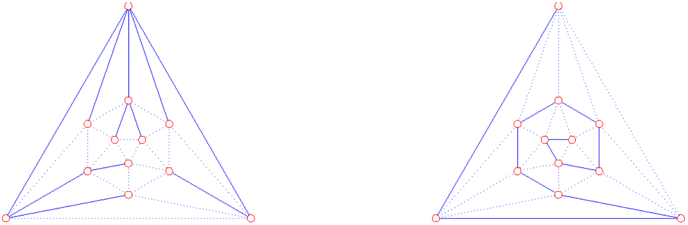

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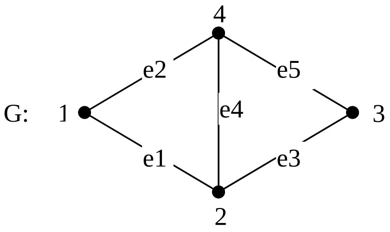
- Cayley's Theorem: $k(K_n) = n^{n-2}$

Incidence matrix ∂_1 and Graph Laplacian $L(G)$



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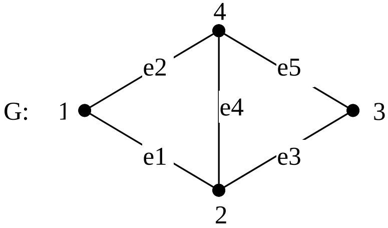

$$\partial_1 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

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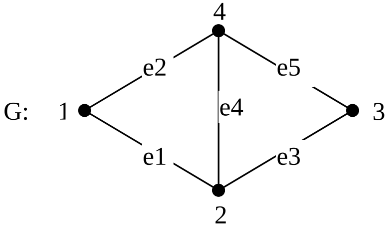

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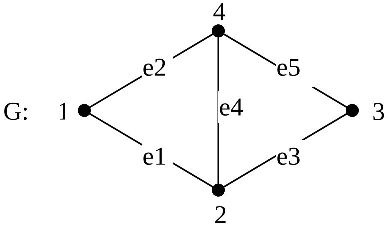

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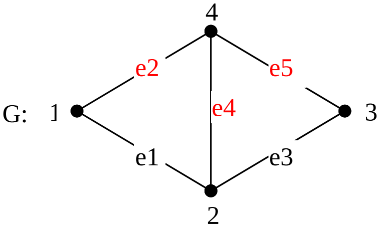

Computing tree numbers

- Matrix-Tree Theorem (Kirchhoff 1847)

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- Temperley's tree-number formula (Temperley 1964)

  $$\det(L(G) + J) = n^2 k(G) \quad (J = \text{all 1's matrix})$$

Example: $L(G)$ and $L(G) + J$

$$G = K_5$$

$$L(G) = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$$

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always singular

non-singular for connected G

Temperley's Formula (1964)

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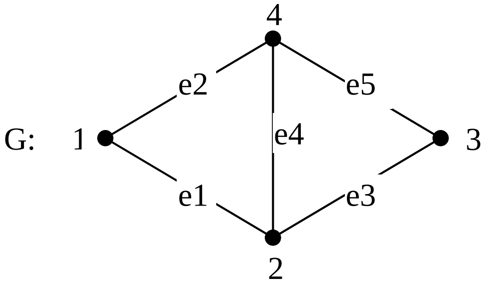
Computing $k(G)$ via ∂_2



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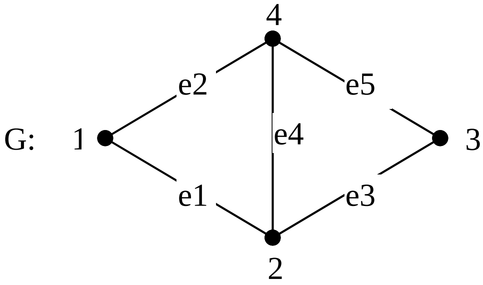

$$\partial_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

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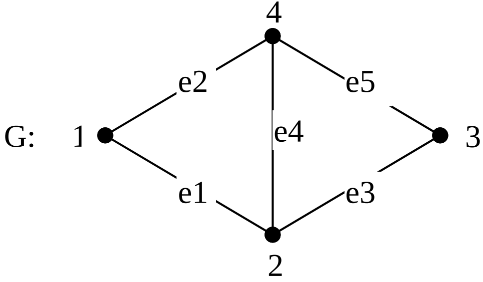

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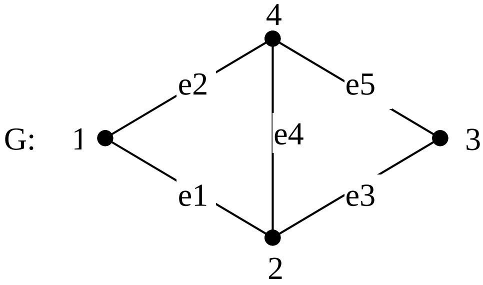

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$$\partial_2^t \partial_2 = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$ cycle-intersection matrix

Exact chain complex for G

G a connected graph with n vertices and m edges

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$$C_2 \simeq \mathbb{Z}^{m-n+1} \xrightarrow{\partial_2} C_1 \simeq \mathbb{Z}^m \xrightarrow{\partial_1} C_0 \simeq \mathbb{Z}^n \xrightarrow{\partial_0} C_{-1} \simeq \mathbb{Z}$$

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- Example: Δ_i for a graph G

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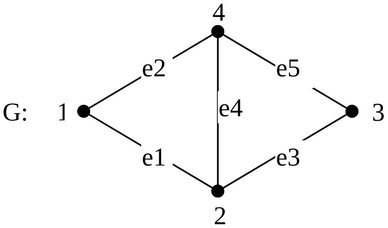
$$\tilde{H}_1(\Gamma_B) = \mathbb{Z}_2 \quad \text{and} \quad w(B) = 2$$

$$\partial_1 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$



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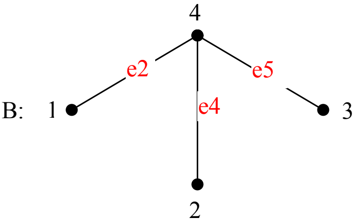

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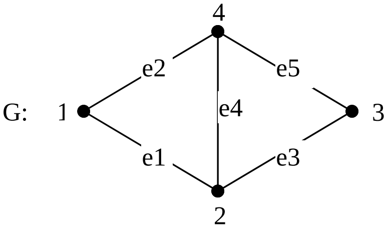

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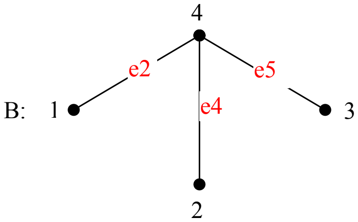

For ∂_i, let $\partial_{\bar{A},B}$ be the submatrix of ∂_i obtained by deleting rows indexed by A and choosing columns indexed by B

Non-singular submatrices of ∂_i

Lemma

Let $r_i = rk\,\partial_i$ *for* $0 \le i \le \dim \Gamma$.
The set of all $r_i \times r_i$ *non-singular submatrices of* ∂_i *is*

$$\mathcal{B}(\partial_i) := \{\, \partial_{\bar{A},B} \mid A \in \mathcal{B}_{i-1} \text{ and } B \in \mathcal{B}_i \,\},$$

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(3) Since Γ is acyclic, $k_d = 1$. Hence,

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Logarithmic tree numbers for acyclic complexes

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Let Γ *be an acyclic complex of dimension* d. *Let*

$$\omega_i = \log \det \Delta_i \quad \textit{and} \quad \kappa_i = \log k_i\,.$$

Define $D(x) = \sum_{i=-1}^{d} \omega_i x^{i+1}$ *and* $K(x) = \sum_{i=0}^{d-1} \kappa_i x^i$.

$$D(x) = (1+x)^2 K(x)\,.$$

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$$\alpha(K_n) = \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2}{2k+1} \frac{(2k+1)!}{2^k \cdot k!} (n-1)^{n-(2k+3)} \qquad (n \geq 2).$$

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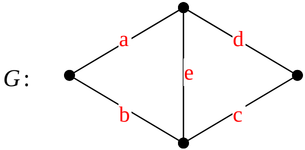
$\mu(W/V) := |\mu(V, W)|$ Möbius function on $L(M) \times L(M)$

Example: augmented matroid complex



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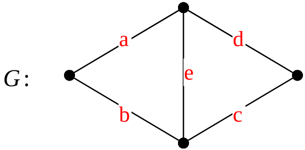

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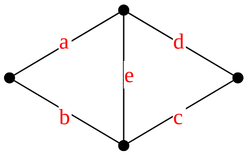

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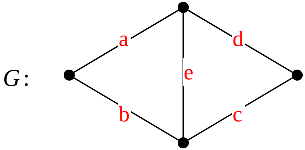

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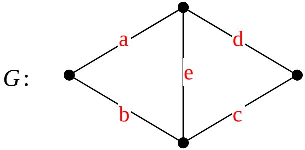

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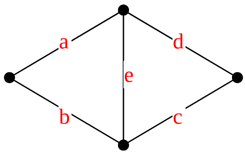

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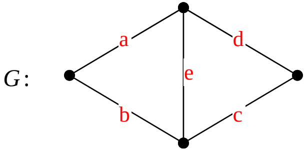

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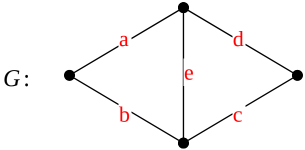

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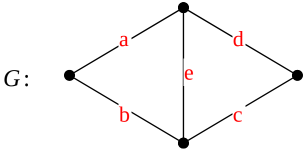

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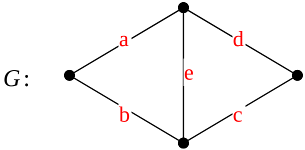

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Question: Find a formula for the multiplicity of $\lambda\ (= 2, 5)$ in k_i.

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(obtained from $\quad K(x) = (1+x)^{-2} D(x)$)

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Question: What is $(-1)^{r(M)} \sum_{W \in L(M)} \mu(\hat{0}, W)\, r(W)$?

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Theorem (High-dimensional tree numbers of a matroid, 2015)

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  Topological data analysis: Vietoris-Rips complex for point cloud

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  High-dimensional tree numbers for acyclic complexes

THANK YOU!

Corollary

Let Γ *be an acyclic complex of dimension* d*. Then*

$$\sum_{q=0}^{d}(-1)^{q+1}q\log\det\Delta_q = 0\,.$$