Giant component, the k-core, and branching processes

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Outline

I. Giant Component of Binomial Random Graph $G(n, p)$

II. Branching Process for Giant Component

III. k-Core of Binomial Random Graph $G(n, p)$

IV. Branching Process for k-Core

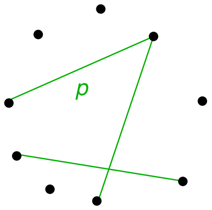
V. Proof Ideas using Warning Propagation

Part I

The Giant Component

in Binomial Random Graph $G(n, p)$

a graph on n vertices, in which each pair of vertices is joined by an edge with probability p, independently of each other

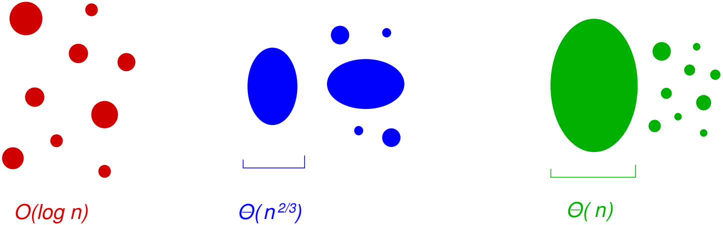


Emergence of Giant Component

$L(d) = \#$ vertices in the largest component in $G(n,p)$ where $d := p\,n$.

Theorem [ERDŐS–RÉNYI 60]

- If $d < 1$, whp $\quad L(d) = O(\log n)$
- If $d = 1$, whp $\quad L(d) = \Theta(n^{2/3})$
- If $d > 1$, whp $\quad L(d) = (1 + o(1))\,\rho\, n \quad$ with $1 - \rho = \exp(-d\rho)$



Strong Law of Large Numbers

$L(d) = \#$ vertices in the largest component in $G(n, p)$ where $d := p\,n > 1$.

Let $\rho \in (0, 1)$ be the unique positive solution of $1 - \rho = \exp(-d\rho)$.

Theorem

For any $\delta, \epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$1 - \epsilon \leq \mathbb{P}\left[\left| \frac{L(d) - \rho\, n}{n} \right| \leq \delta \right] \leq 1 + \epsilon.$$

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Furthermore, we have

$$\mathbb{E}[L(d)] = \rho\, n =: \mu$$
$$\mathbb{V}[L(d)] = \frac{\rho(1-\rho)}{(1-d(1-\rho))^2}\, n =: \sigma^2$$

Central Limit Theorem

$L(d) = \#$ vertices in the largest component in $G(n,p)$ where $d := p\,n > 1$.

Let $\mu := \rho\, n$ and $\sigma^2 := \frac{\rho(1-\rho)}{(1-d(1-\rho))^2}\, n$ where $1-\rho = \exp(-d\rho)$.

Theorem [BEHRISCH–COJA-OGHLAN–K. 09]

For any $a, b \in \mathbb{R}$ with $a < b$ and any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$(1-\delta)\, F(a,b) \leq \mathbb{P}\left[a \leq \frac{L(d)-\mu}{\sigma} \leq b \right] \leq (1+\delta)\, F(a,b),$$

where

$$F(a,b) := \frac{1}{\sqrt{2\pi}} \int_a^b \exp\left(-\frac{x^2}{2}\right)\, dx.$$

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$\Rightarrow$ It provides the estimation of $L(d)$ up to an error of $o(\sigma) = o(\sqrt{n})$.

Local Limit Theorem for Giant Component

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For any compact interval $I \subset \mathbb{R}$ and any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and any $s \in \mathbb{N}$ satisfying $n^{-1/2}(s-\mu) \in I$

$$(1-\delta)\, f(s) \leq \mathbb{P}[\, L(d) = s\,] \leq (1+\delta)\, f(s),$$

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$\Rightarrow$ It provides the asymptotic probability that $L(d)$ hits exactly the value s.

Local Limit Theorem for Joint Distribution

$L(d) \ = \ \#$ vertices

$M(d) \ = \ \#$ edges in the largest component in $G(n,p)$ where $d := p\,n > 1$.

Theorem [Behrisch–Coja-Oghlan–K. 14]

For any compact set $\mathcal{J} \subset \mathbb{R}^2$ and any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and any $s, t \in \mathbb{N}$ satisfying $(n^{-1/2}(s-\mu), n^{-1/2}(t-\tilde{\mu})) \in \mathcal{J}$

$$(1-\delta)\, P(s,t) \ \leq \ \mathbb{P}[\ L(d) \ = \ s \ \text{ and } M(d) \ = \ t\ ] \ \leq \ (1+\delta)\, P(s,t),$$

$$P(s,t) := \frac{1}{2\pi\sqrt{\sigma^2\tilde{\sigma}^2 - \hat{\sigma}^2}} \ \exp\left(-\frac{\sigma^2\tilde{\sigma}^2}{2(\sigma^2\tilde{\sigma}^2-\hat{\sigma}^2)}\left(\frac{(s-\mu)^2}{\sigma^2} - \frac{2\hat{\sigma}(s-\mu)(t-\tilde{\mu})}{\sigma^2\tilde{\sigma}^2} + \frac{(t-\tilde{\mu})^2}{\tilde{\sigma}^2}\right)\right),$$

$$\mu := \rho\, n, \quad \tilde{\mu} := \frac{d(1-(1-\rho)^2)}{2}\, n, \quad \sigma^2 := \frac{\rho(1-\rho)}{(1-d(1-\rho))^2}\, n,$$

$$\hat{\sigma} := \frac{d(1-\rho)\left(1-(1-\rho)^2 - d\rho(1-\rho)\right)}{(1-d(1-\rho))^2}\, n,$$

$$\tilde{\sigma}^2 := \left(\frac{d^2(1-\rho)^2\,(2+(1-\rho)\,(-2d\rho-2-\rho))}{(1-d(1-\rho))^2} + \frac{d(1-(1-\rho)^2)}{2}\right) n.$$

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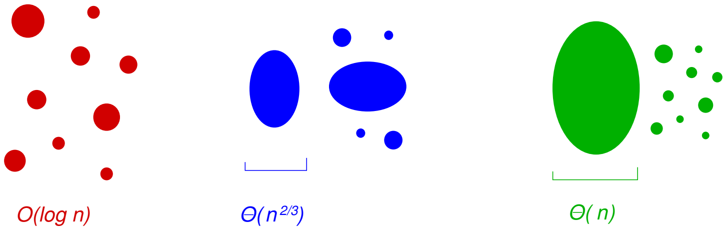


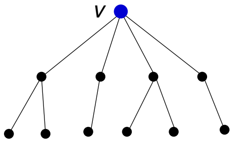
Part II

Branching Processes

for the Giant Component in $G(n, p)$ where $d := p\,n$

Component-Exposure Process

(1) Breadth-first search



Construct spanning tree T_v

of component C_v that contains vertex v

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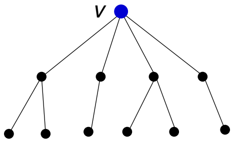

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(2) Coupling with Galton-Watson branching process

with offspring disbribution $\mathrm{Po}(\mathrm{d})$

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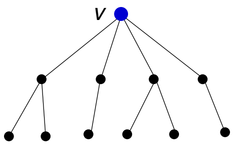

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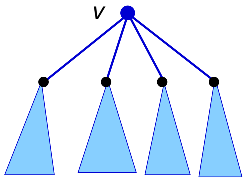

It starts with a single vertex v and

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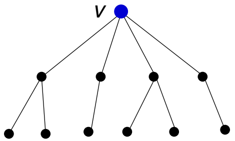

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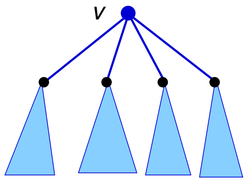


It starts with a single vertex v and

the number of children of each vertex

is an i.i.d random variable

with distribution $\mathrm{Po}(\mathrm{d})$

Galton-Watson Branching Process

The number of children of each vertex in the Galton-Watson process with offspring distribution $\mathrm{Po}(\mathrm{d})$ is given by the probability generating function

$$
\begin{aligned}
g(x) &= \sum_{n\geq 0} \mathbb{P}\,[\text{ a vertex generates } n \text{ offspring }]\; x^n \\
&= \sum_{n\geq 0} \mathbb{P}\,[\mathrm{Po}(\mathrm{d}) = n]\; x^n \\
&= \sum_{n\geq 0} \exp(-d)\, \frac{d^n}{n!}\; x^n \\
&= \exp(-d) \sum_{n\geq 0} \frac{(dx)^n}{n!} \\
&= \exp(-d(1-x))
\end{aligned}
$$

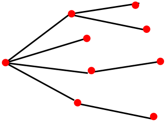
i.e. $\mathbb{P}\,[\text{ a vertex generates } n \text{ offspring }] = \exp(-d)\, \dfrac{d^n}{n!} =: [x^n]\, g(x)$

Galton-Watson Branching Process

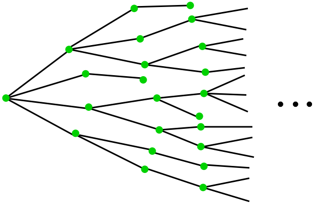
Theorem

The number of children is given by i.i.d. random variable $\sim \mathrm{Po(d)}$.

- If $d < 1$, the process dies out with probability 1.
- If $d > 1$, with positive probability ρ the process continues forever.



„small" component in $G(n, p)$



„giant" component of size $\rho n + o(n)$

in $G(n, p)$ where $1 - \rho = \exp(-d\rho)$

Extinction Probability of Galton-Watson Process

Let T be the total number of vertices created in the process. Suppose $d > 1$.

Consider the probability generating function for T

$$p(x) := \sum_i \mathbb{P}(T = i)\, x^i.$$

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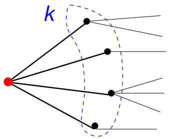
It satisfies

$$\begin{aligned} p(x) &= x \sum_k \mathbb{P}(\mathrm{Po(d)} = k)\, p(x)^k \\ &= x \sum_k \frac{\exp(-d) d^k}{k!} p(x)^k = x \exp(-d\,(1 - p(x))). \end{aligned}$$



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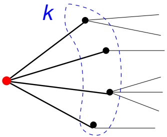

Because $\rho = 1 - p(1)$, we have $1 - \rho = \exp(-d\rho)$.

Part III

The k-Core $\mathcal{C}_k(\boldsymbol{G})$

of Binomial Random Graph $\mathbf{G} = \mathbf{G}(\mathbf{n}, \mathbf{p})$

where $\mathbf{d} := \mathbf{p}\,\mathbf{n}$ constant and $\mathbf{k} \geq \mathbf{3}$

is the maximal subgraph of $\boldsymbol{G}$ of minimum degree k

The k-core $\mathcal{C}_k(\boldsymbol{G})$ of $\boldsymbol{G} = \boldsymbol{G}(n, \frac{d}{n})$

Theorem [Pittel–Spencer–Wormald 96]

Let
$$d_k \;:=\; \inf\left\{\frac{\lambda}{\mathbb{P}\left[\mathrm{Po}(\lambda) \geq k-1\right]} : \lambda > 0\right\}.$$

For any $d > d_k$ the equation $d = \lambda(\mathbb{P}\left[\mathrm{Po}(\lambda) \geq k-1\right])^{-1}$ has precisely two solutions and let $\lambda_k(d)$ denote the larger one. Define

$$\psi_k : (d_k, \infty) \to (0, \infty), \qquad d \mapsto \mathbb{P}\left[\mathrm{Po}(\lambda_k(d)) \geq k\right].$$

Then ψ_k is a strictly increasing continuous function.

- If $d < d_k$, then whp $\mathcal{C}_k(\boldsymbol{G}) = \emptyset$.
- If $d > d_k$, then $\frac{1}{n}|\mathcal{C}_k(\boldsymbol{G})|$ converges in probability to $\psi_k(d)$.

Fixed Point p^*

Theorem [PITTEL–SPENCER–WORMALD 96]

If $d > d_k$, then $\frac{1}{n}|\mathcal{C}_k(\boldsymbol{G})|$ converges in probability to $\psi_k(d)$.

Lemma [COJA-OGHLAN–COOLEY–KANG–SKUBCH 15+]

Suppose $d > d_k$ and let p^* be the largest fixed point of the function

$$\phi_{d,k} : [0,1] \to [0,1], \qquad p \mapsto \mathbb{P}[\,\mathrm{Po}(dp) \geq k-1\,].$$

Then $\phi_{d,k}$ is contracting on $[p^*, 1]$.

Moreover,

$$\psi_k(d) = \phi_{d,k+1}(p^*) = \mathbb{P}[\,\mathrm{Po}(dp^*) \geq k\,].$$

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$\Rightarrow$ For $d > d_k$, $\frac{1}{n}|\mathcal{C}_k(\boldsymbol{G})|$ converges in probability to $\mathbb{P}\,[\,\mathrm{Po}(dp^*) \geq k\,]$.

Part IV

Branching Processes

for the k-Core $\mathcal{C}_k(\boldsymbol{G})$

of $\mathbf{G} = \mathbf{G}(\mathbf{n}, \mathbf{p})$ where $\mathbf{d} := \mathbf{p}\,\mathbf{n} > \mathbf{d_k}$ and $\mathbf{k} \geq 3$

5-Type Branching Process for the k-Core

- It starts with a single vertex v_0 of type $\boldsymbol{t} \in \{000, 010, 110\}$ such that

$$
\begin{aligned}
\mathbb{P}[v_0 \text{ is of type } 000] &= \mathbb{P}(\mathrm{Po}(dp^*) < k-1) \\
\mathbb{P}[v_0 \text{ is of type } 010] &= \mathbb{P}(\mathrm{Po}(dp^*) = k-1) \\
\mathbb{P}[v_0 \text{ is of type } 110] &= \mathbb{P}(\mathrm{Po}(dp^*) \geq k),
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where $$p^* = \mathbb{P}(\mathrm{Po}(dp^*) \geq k-1).$$

- Subsequently, each vertex of type $\boldsymbol{t} \in \{000, 001, 010, 110, 111\}$ generates a random number of vertices of each type such that for an integer vector $\boldsymbol{n} = (n_{000}, n_{001}, n_{010}, n_{110}, n_{111})$,

$$\mathbb{P}[\text{a vertex of type } \boldsymbol{t} \text{ generates offspring } \boldsymbol{n}] = [x_{000}^{n_{000}} \cdots x_{111}^{n_{111}}]\, g_{\boldsymbol{t}}(\boldsymbol{x}),$$

where $\boldsymbol{x} = (x_{000}, x_{001}, x_{010}, x_{110}, x_{111})$, and $g_{\boldsymbol{t}}(\boldsymbol{x})$ are given below.

Probability Generating Functions $g_{\boldsymbol{t}}(\boldsymbol{x})$

$$
\begin{aligned}
g_{000}(\boldsymbol{x}) &:= \exp(d(1-p^*)(x_{000}-1))\frac{\sum_{h=0}^{k-2}(dp^*)^h(qx_{010}+(1-q)x_{110})^h/h!}{\sum_{h=0}^{k-2}(dp^*)^h/h!} \\
g_{001}(\boldsymbol{x}) &:= \bar{q}\left(\exp(d(1-p^*)(x_{001}-1))\,(qx_{010}+(1-q)x_{110})^{k-2}\right) \\
&\quad + (1-\bar{q})\left(\exp(d(1-p^*)(x_{000}-1))\frac{\sum_{h=0}^{k-3}(dp^*)^h(qx_{010}+(1-q)x_{110})^h/h!}{\sum_{h=0}^{k-3}(dp^*)^h/h!}\right) \\
g_{010}(\boldsymbol{x}) &:= \exp(d(1-p^*)(x_{001}-1))\,(qx_{010}+(1-q)x_{110})^{k-1} \\
g_{110}(\boldsymbol{x}) &:= \exp(d(1-p^*)(x_{001}-1))\frac{\sum_{h\geq k}(dp^*x_{111})^h/h!}{\sum_{h\geq k}(dp^*)^h/h!} \\
g_{111}(\boldsymbol{x}) &:= \exp(d(1-p^*)(x_{001}-1))\frac{\sum_{h\geq k-1}(dp^*x_{111})^h/h!}{\sum_{h\geq k-1}(dp^*)^h/h!}
\end{aligned}
$$

where

$$
\begin{aligned}
q = q(d,k,p^*) &:= \mathbb{P}\left[\mathrm{Po}(dp^*) = k-1 \mid \mathrm{Po}(dp^*) \geq k-1\right] \\
\bar{q} = \bar{q}(d,k,p^*) &:= \mathbb{P}\left[\mathrm{Po}(dp^*) = k-2 \mid \mathrm{Po}(dp^*) \leq k-2\right].
\end{aligned}
$$

$\{0,1\}$-marked Random Rooted Tree

- Let $\widehat{\boldsymbol{T}}(d,k,p^*)$ denote the 5-type random tree rooted at v_0 constructed by the 5-type branching process.
- Given $\widehat{\boldsymbol{T}}(d,k,p^*)$, give mark 0 to all vertices of type 000, 001 or 010, and mark 1 to vertices of type 110 or 111.
- Let $\boldsymbol{T}(d,k,p^*)$ signify the resulting $\{0,1\}$-marked random rooted tree.

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How is $\boldsymbol{T}(d,k,p^*)$ related to $\mathcal{C}_k(\boldsymbol{G})$?

$\{0,1\}$-marked Rooted Graphs

Definition

- A $\{0,1\}$-marked rooted graph is a locally finite connected graph G with a distinguished vertex $v_0 \in V(G)$ and a map $\sigma : V(G) \to \{0,1\}$.
- Two $\{0,1\}$-marked rooted graphs (G, v_0, σ), (G', v_0', σ') are *isomorphic* if $\exists$ an isomorphism $\pi : G \to G'$ such that $\pi(v_0) = v_0'$, $\sigma = \sigma' \circ \pi$.
- For $s \in \mathbb{N}$ let $\partial^s[G, v_0, \sigma]$ be the isomorphism class of the $\{0,1\}$-marked rooted graph obtained by deleting all vertices at distance $> s$ from v_0.

$\mathcal{C}_k(\boldsymbol{G})$ vs $\boldsymbol{T}(d,k,p^*)$

Theorem [Coja-Oghlan–Cooley–Kang–Skubch 15+]

Let $\boldsymbol{G} = \boldsymbol{G}(n,p)$ with $d := pn > d_k$.

For $v \in V(\boldsymbol{G})$, let $\boldsymbol{G}_v$ denote the component of v. Define

$$\sigma_{k,\boldsymbol{G}_v} : V(\boldsymbol{G}_v) \to \{0,1\}, \qquad v \mapsto \mathbf{1}\{v \in \mathcal{C}_k(\boldsymbol{G})\}.$$

Let $s \in \mathbb{N}$ and τ be a $\{0,1\}$-marked rooted tree. Then

$$\frac{1}{n}\sum_{v \in V(\boldsymbol{G})} \mathbf{1}\left\{ \partial^s[\,\boldsymbol{G}_v, v, \sigma_{k,\boldsymbol{G}_v}\,] = \partial^s[\tau] \right\}$$

converges in probability to

$$\mathbb{P}\left[\,\partial^s[\,\boldsymbol{T}(d,k,p^*)\,] = \partial^s[\tau]\,\right].$$

Part V

Proof Ideas

using Warning Propagation

Warning Propagation

[MÉZARD–MONTANARI 09]

Given a graph G

we introduce "messages" on $E(G)$ and marks on $V(G)$ as follow.

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- At time $t = 0$ we start with $\mu_{v\to u}(0|G) = 1$ for all $\{u, v\} \in E(G)$.

  For each $v \in V(G)$ let $N(v)$ denote the neighbourhood of v.

  For $t \geq 1$, set

  $$\mu_{v\to u}(t|G) = \mathbf{1}\left\{\left(\sum_{w\in N(v)\setminus\{u\}} \mu_{w\to v}(t-1|G)\right) \geq k-1\right\}$$

  for $\{u, v\} \in E(G)$. The messages are "directed".

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Warning Propagation and the k-core

Lemma [Coja-Oghlan–Cooley–Kang–Skubch 15+]

Let G be a locally finite graph.

- For any vertex $v \in V(G)$, $\lim_{t\to\infty} \mu_v(t|G)$ exists.
- $v \in \mathcal{C}_k(G) \iff \lim_{t\to\infty} \mu_v(t|G) = 1$.

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Analysis of Warning Propagation fixed point on random graph $\boldsymbol{G} = \boldsymbol{G}(n, p)$ with $d := pn > d_k$ reduces to the study of Warning Propagation on the (single-type) Galton-Watson tree $\boldsymbol{T} = \boldsymbol{T}(d)$ with $\mathrm{Po}(\mathrm{d})$ offspring.

Triple-Marks on Vertices of $\boldsymbol{T} = \boldsymbol{T}(d)$

Primary mark

$$\mu_v(t|\boldsymbol{T}) = \mathbf{1}\left\{\left(\mu_{\downarrow v}(t|\boldsymbol{T}) + \sum_{w \in N_+(v)} \mu_{w\uparrow}(t|\boldsymbol{T})\right) \geq k\right\}$$

Bottom-up message

$$\mu_{v\uparrow}(t|\boldsymbol{T}) = \mathbf{1}\left\{\left(\sum_{w \in N_+(v)} \mu_{w \to v}(t-1|\boldsymbol{T})\right) \geq k-1\right\}$$

Top-down message

$$\mu_{\downarrow v}(t|\boldsymbol{T}) = \mathbf{1}\left\{\left(\mu_{\downarrow u}(t-1|\boldsymbol{T}) + \sum_{w \in N_+(u) \setminus \{v\}} \mu_{w\uparrow}(t-1|\boldsymbol{T})\right) \geq k-1\right\}$$

Mark v with the triple $(\mu_v(t|\boldsymbol{T}), \mu_{v\uparrow}(t|\boldsymbol{T}), \mu_{\downarrow v}(t|\boldsymbol{T}))$ and denote by $\widehat{\boldsymbol{T}}_t(d,k)$ the resulting $\{000, 001, 010, 110, 111\}$-marked random rooted tree.

Distribution of $\partial^s[\widehat{\boldsymbol{T}}_t(d,k)]$ and $\partial^s[\widehat{\boldsymbol{T}}(d,k,p^*)]$

Let $\widehat{\boldsymbol{T}}_t(d,k)$ denote the $\{000, 001, 010, 110, 111\}$-marked random rooted tree whose vertices are marked with $(\mu_v(t|\boldsymbol{T})), \mu_{v\uparrow}(t|\boldsymbol{T})), \mu_{\downarrow v}(t|\boldsymbol{T}))$.

Let $\widehat{\boldsymbol{T}}(d,k,p^*)$ denote the 5-type random rooted tree constructed by the 5-type branching process.

Theorem [Coja-Oghlan–Cooley–Kang–Skubch 15+]

For any $s \geq 0$

$$\lim_{t\to\infty} \mathcal{L}(\partial^s[\widehat{\boldsymbol{T}}_t(d,k)]) = \mathcal{L}(\partial^s[\widehat{\boldsymbol{T}}(d,k,p^*)]).$$

Derivation of Distribution of Root Type

- $\mu^*_{\downarrow v_0} = 0$ with certainty.
- $\mu^*_{v_0\uparrow}$ has distribution $\mathrm{Be}(p^*)$.
- $\mu^*_{v_0} = 0$ if $\mu^*_{v_0\uparrow} = 0$.
- Conditioned on that $\mu^*_{v_0\uparrow} = 1$,

  $\sum_{w\in\partial v_0} \mu^*_{w\uparrow}$ has distribution $\mathrm{Po}_{\geq k-1}(dp^*)$, and

  $\mu^*_{v_0} = 1$ iff $\sum_{w\in\partial v_0} \mu^*_{w\uparrow} \geq k$.
- Using $q = \mathbb{P}\left[\mathrm{Po}(dp^*) = k-1 | \mathrm{Po}(dp^*) \geq k-1\right]$ and

  the fixed point property $p^* = \mathbb{P}\left(\mathrm{Po}(dp^*) \geq k-1\right)$, we obtain

$$
\begin{aligned}
\mathbb{P}\left((\mu^*_{v_0}, \mu^*_{v_0\uparrow}, \mu^*_{\downarrow v_0}) = 000\right) &= 1 - p^* &= \mathbb{P}\left(\mathrm{Po}(dp^*) < k-1\right) \\
\mathbb{P}\left((\mu^*_{v_0}, \mu^*_{v_0\uparrow}, \mu^*_{\downarrow v_0}) = 010\right) &= p^* q &= \mathbb{P}\left(\mathrm{Po}(dp^*) = k-1\right) \\
\mathbb{P}\left((\mu^*_{v_0}, \mu^*_{v_0\uparrow}, \mu^*_{\downarrow v_0}) = 110\right) &= p^*(1-q) &= \mathbb{P}\left(\mathrm{Po}(dp^*) \geq k\right).
\end{aligned}
$$

Derivation of the Generating Functions

- Let $\tau_v(t_2)= \#$ messages of type t_2 that v receives from its children.

Suppose $(\mu^*_v, \mu^*_{v\uparrow}, \mu^*_{\downarrow v}) = 000$.

- $\tau_v(0)$ has distribution $\mathrm{Po}(d(1-p^*))$
- $\mu^*_{\downarrow w} = 0$ for all children w of v.
- Since $\mu^*_{v\uparrow} = 0$, $\tau_v(1) \le k-2$, so $\tau_v(1)$ has distribution $\mathrm{Po}_{\le k-2}(dp^*)$.
- For a child w of v, conditioned on $\mu^*_{w\uparrow} = 1$,

  $\mu^*_w = 1$ iff w has at least k children y with $\mu^*_{y\uparrow} = 1$.

In summary, we obtain the generating function

$$g_{000}(\boldsymbol{x}) = \exp(d(1-p^*)x_{000}) \frac{\sum_{h=0}^{k-2} (dp^*)^h (qx_{010} + (1-q)x_{110})^h/h!}{\sum_{h=0}^{k-2}(dp^*)^h/h!}.$$

Summary

Erdős-Rényi random graph $G(n, p)$ where $d = p\,n$.

(1) The giant component

– emerges when $d > 1$

– is of order $(1 + o(1))\rho n$ where $1 - \rho = \exp(-d\rho)$

(2) The k-core (for $k \geq 3$)

– is the largest subgraph of minimum degree k

– emerges when $d > d_k$

– its scaled size converges in probability to $\mathbb{P}\,[\mathrm{Po}(dp^*) \geq k]$ where $p^* = \mathbb{P}\,[\mathrm{Po}(dp^*) \geq k - 1]$.

(3) Branching processes

– Galton-Watson BP with Poisson $\mathrm{Po}(\mathrm{d})$ offspring distribution

– $\{0, 1\}$-marked random tree $\boldsymbol{T}(d, k, p^*)$ constructed by 5-type branching process, whose mark indicate membership of the k-core